University of Crete
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Master's Thesis

Decay of Circular Means of Fourier Transform of Measures

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INTRODUCTION

Steinhaus's theorem asserts that if $A \subseteq \mathbb{R}$ has positive Lebesgue measure, then $A - A$ contains a neighborhood of the origin. Although both the statement and the proof of this result are elementary, its higher-dimensional analogue has become one of the central difficult problems in geometric measure theory: Falconer's distance set conjecture. The conjecture states that if $A \subseteq \mathbb{R}^2$ has Hausdorff dimension greater than 1, then its distance set

$$\Delta A = \{|x - y| : x \in A,\ y \in A\}$$

has positive Lebesgue measure.

Pertti Mattila was the first to attack this problem using methods from harmonic analysis and geometric measure theory. His starting point was the following observation. Let μ be a measure supported on A, and let ν_μ be its push-forward under the distance map:

$$\nu_\mu(B) = \mu \times \mu \left\{(x, y) : |x - y| \in B\right\}, \qquad B \subset \mathbb{R} \text{ Borel.}$$

If the Fourier transform of ν_μ belongs to L^2, then $|\Delta A| > 0$. Mattila's method therefore leads naturally to estimates for spherical averages of Fourier transforms of measures. The theorem analyzed in this thesis, due to Thomas Wolff, gives an estimate for circular averages of Fourier transforms of measures and remains the strongest result known for the planar distance set problem.

Keywords. Harmonic analysis, geometric measure theory, "distance set conjecture," Hausdorff measure.

Abstract

This thesis studies circular averages of Fourier transforms of measures in connection with Falconer's distance set conjecture. If $A \subseteq \mathbb{R}^2$ has Hausdorff dimension greater than 1, the conjecture predicts that its distance set

$$\Delta A = \{|x-y| : x \in A,\ y \in A\}$$

has positive Lebesgue measure.

Mattila's approach relates this question to the Fourier transform of the push-forward of a measure under the distance map. In particular, if μ is supported on A and the associated distance measure ν_μ satisfies $\widehat{\nu_\mu} \in L^2$, then $|\Delta A| > 0$. This reduces the problem to estimating spherical, and in the planar case circular, averages of Fourier transforms of measures.

The main result analyzed here is a theorem of Thomas Wolff giving an upper bound for such circular averages. As a consequence, one obtains the best currently known progress toward Falconer's conjecture in the plane.

Keywords. Harmonic analysis, geometric measure theory, distance set conjecture, Hausdorff measure.

THE THEOREM AND ITS PROOF

We begin by showing how circular averages are connected with the distance set conjecture. For this purpose, we take the following facts as known.

(I) The Hausdorff dimension of a compact set E coincides with the number

$$\sup\{a: \text{ there exists a probability measure } \mu \text{ on } E \text{ with } I_a(\mu) < +\infty\}$$

(II)

$$\int\int |x-y|^{-\alpha} d\mu(x) d\mu(y) = c_\alpha \int |\widehat{\mu}(\xi)|^2 |\xi|^{\alpha-n} d\,\xi$$

(III) If σ_R is the surface measure on the circle centered at 0 with radius R, then

$$\widehat{\sigma_R} = 2(R|x|)^{\frac{1}{2}} \cos\bigl(2\pi(R|x| - \tfrac{1}{8})\bigr) + \Omega_R(x)$$

where $\Omega_R(x) = \mathcal{O}\bigl((R|x|)^{-\frac{3}{2}}\bigr)$ for $R|x| > 1$ and $\Omega_R(x) = \mathcal{O}\bigl((R|x|)^{-\frac{1}{2}}\bigr)$ for $R|x| < 1$.

(IV) If $\widehat{\mu} \in L^2$, then $\mu = f\,dx$ for some $f \in L^2$.

Let $E \subseteq \mathbb{R}^2$ with $\dim_{\mathcal{H}} E = \alpha$. By (I), there exists a probability measure μ on E with finite α-dimensional energy. Let ν_μ be the push-forward of μ to the distance set of E, namely $\Delta E = \{|x-y| : x, y \in E\}$. Thus

$$d\nu_\mu(t) = |x-y| d\mu(x) d\mu(y), \quad (t = |x-y|)$$

The measure ν_μ is supported on ΔE, and $I_\alpha(\mu) = \int t^{-\alpha}\, d\nu_\mu$. If we show that $\widehat{\nu_\mu} \in L^2$, then by (IV) there exists $f \in L^2$ with $d\nu_\mu(t) = f\,dt$, and since $I_\alpha(\mu) > 0$ it follows that $|\Delta E| > 0$. For technical reasons, we shall instead work with the measure

$$\nu = d\nu(t) = e^{i\frac{\pi}{4}} t^{-\frac{1}{2}} d\nu_\mu(t) + e^{-i\frac{\pi}{4}} |t|^{-\frac{1}{2}} d\nu_\mu(-t),$$

which is supported on $\Delta E \cup -\Delta E$.

Lemma. *The following are equivalent:*

(1) $\widehat{\nu} \in L_2(\mathbb{R})$,

(2)

$$\int_{R=1}^{+\infty} \left(\int_{-\pi}^{\pi} \left|\widehat{\mu}\left(Re^{i\theta}\right)\right|^2 d\,\theta \right)^2 R\,dR \le \infty.$$

Proof.

$$\begin{aligned}\widehat{\nu}(k) &= e^{i\frac{\pi}{4}} \int |x-y|^{-\frac{1}{2}} e^{-2i\pi k|x-y|}\, d\mu(x) d\,\mu(y) + e^{-i\frac{\pi}{4}} \int |x-y|^{-\frac{1}{2}} e^{2i\pi k|x-y|}\, d\mu(x) d\,\mu(y) \\ &= 2\int |x-y|^{-\frac{1}{2}} \cos(2\pi(k|x-y| - \tfrac{1}{8})) d\mu(x)\, d\mu(y)\end{aligned}$$

On the other hand, for $k > 0$,

$$\begin{aligned}
\int_{-\pi}^{\pi} \left|\widehat{\mu}\left(ke^{i\theta}\right)\right|^2 d\,\theta &= \int_{\mathbb{R}^2} \widehat{\sigma_k} * \mu \, d\mu \\
&= 2k^{-\frac{1}{2}} \iint |x-y|^{-\frac{1}{2}} \cos(2\pi k|x-y| - \tfrac{1}{8})\, d\mu(x)\, d\mu(y) + \iint \Omega(k|x-y|)\, d\mu(x)\, d\mu(y) \\
&= 2k^{-\frac{1}{2}} \iint |x-y|^{-\frac{1}{2}} \cos(2\pi k|x-y| - \tfrac{1}{8})\, d\mu(x)\, d\mu(y) \\
&+ \mathcal{O}\left(\iint\limits_{k|x-y|>1} (k|x-y|)^{-\frac{3}{2}}\, d\mu(x)\, d\mu(y) \right) \\
&+ \mathcal{O}\left(\iint\limits_{k|x-y|\le 1} (k|x-y|)^{-\frac{1}{2}}\, d\mu(x)\, d\mu(y) \right) \\
&= 2k^{-\frac{1}{2}} \iint |x-y|^{-\frac{1}{2}} \cos(2\pi k|x-y| - \tfrac{1}{8})\, d\mu(x)\, d\mu(y) + \mathcal{O}\left(\iint (k|x-y|)^{-\alpha}\, d\mu(x)\, d\mu(y) \right)
\end{aligned}$$

for every $\alpha \in \left[\frac{1}{2}, \frac{3}{2}\right]$.

Therefore,

$$\widehat{\nu}(k) = |k|^{\frac{1}{2}} \int_{-\pi}^{\pi} \left|\widehat{\mu}\left(ke^{i\theta}\right)\right|^2 d\,\theta + \mathcal{O}\big(|k|^{\frac{1}{2}-\alpha} I_\alpha(\mu)\big).$$

The second term belongs to $L^2_{\{|k|>1\}}$. Since $\widehat{\nu} \in L^2([-1,1])$ in any case, we conclude that $\widehat{\nu}(k)$ belongs to L_2 if and only if $|k|^{\frac{1}{2}} \int_{-\pi}^{\pi} \left|\widehat{\mu}\left(ke^{i\theta}\right)\right|^2 d\,\theta$ belongs to L_2. ☐

Proposition. *Let $\alpha > 1$ be such that, whenever μ is a compactly supported measure with finite α-dimensional energy, we have*

$$\int_{-\pi}^{\pi} |\widehat{\mu}(Re^{i\theta})|^2\, d\theta \le C_\mu R^{\alpha-2}.$$

Then every compact subset of $\mathbb{R}^2$ with dimension greater than α has distance set of positive Lebesgue measure.

Proof. Let E be a compact subset of $\mathbb{R}^2$ with $\dim E > \alpha$. By (I), there exists a measure μ supported on E such that $I_\alpha(\mu) < \infty$. From (II) we obtain

$$\int_{R=1}^{+\infty} \left(\int_{-\pi}^{\pi} \left|\widehat{\mu}\left(Re^{i\theta}\right)\right|^2 d\,\theta \right)^2 R\, dR \le C_\mu \int_{R=1}^{+\infty} \int_{-\pi}^{\pi} \left|\widehat{\mu}\left(Re^{i\theta}\right)\right|^2 R^{\alpha-2} R\, dR d\,\theta \le C_{\alpha,\mu} I_\alpha(\mu) \le +\infty.$$

Using Lemma 1, it follows that $\Delta E \cup -\Delta E$ has positive Lebesgue measure, and hence so does ΔE. ☐

Theorem. *Let $\alpha \in (0,2)$. Then for every $\epsilon > 0$ there exists a constant C_ϵ such that the following holds. If μ is a positive measure on $\mathbb{R}^2$ whose support is contained in the unit disk and whose α-dimensional energy* *satisfies*

$$I_\alpha(\mu) \overset{def}{=} \iint \frac{1}{|x-y|^\alpha}\, d\mu(x)d\mu(y) = 1,$$

then for every $R \geq 1$*,*

$$\int_{-\pi}^{\pi} \left|\widehat{\mu}\left(Re^{i\theta}\right)\right|^2\, d\theta \leq C_\epsilon R^{-\frac{\alpha}{2}+\epsilon}.$$

This theorem yields the distance set conjecture for $\alpha > \frac{4}{3}$. Indeed, for every $\epsilon > 0$ and $\alpha = \frac{4}{3} + \epsilon$ we have $R^{\epsilon-\frac{\alpha}{2}} = R^{\alpha-2}$, so the previous proposition applies.

The proof of the theorem uses a number of elementary geometric arguments together with standard properties of the Fourier transform. We first establish several auxiliary propositions and lemmas, then introduce the technical tools needed later, and finally carry out the main argument.

In what follows, when we say that two rectangles have parallel sides, we mean that their long sides are parallel and their short sides are parallel as well. The axis of a rectangle is the line parallel to its long sides. When we speak of the angle between two rectangles, we mean the angle between their axes. If ρ is a rectangle, then a dual rectangle is any rectangle whose axis is perpendicular to that of ρ, whose length equals the reciprocal of the width of ρ, and whose width equals the reciprocal of the length of ρ.

Proposition 1. *Let* C *be a constant and let* $\mathcal{A}$ *be a family of rectangles with length between* l *and* Cl *and width between* w *and* Cw*, such that whenever* $R_1, R_2 \in \mathcal{A}$*, we have* $R_1 \subset CR_2$*. If the cardinality of* $\mathcal{A}$ *is greater than a constant depending only on* C*, then there exist* $R, R' \in \mathcal{A}$ *such that* $R \subset 2R'$*.*

Proof. Fix a rectangle R_0 in $\mathcal{A}$ and take the rectangle NR_0, with length $\leq C^2 l$ and width $\leq C^2 w$, so that it contains all the remaining rectangles in $\mathcal{A}$. Next, for each $R \in \mathcal{A}$ consider $\frac{1}{N}R$. If all the $\frac{1}{N}R$ are pairwise disjoint, then $(\#\mathcal{A})\frac{lw}{N^2} \leq C^4 lw$. Let $p = C^{\frac{1}{N}}$. Partition $[l, Cl]$ into the intervals $[l, pl], [pl, p^2 l], \ldots, [p^{N-1}l, p^N l]$ and $[w, Cw]$ into the intervals $[w, pw], [pw, p^2 w], \ldots, [p^{N-1}w, p^N w]$. If $\#\mathcal{A} \geq (N+1)(N+1)N^2C^4$, then there exist two rectangles R_1, R_2 in $\mathcal{A}$ such that $\frac{1}{N}R_1$, $\frac{1}{N}R_2$ have nonempty intersection, their lengths belong to the same interval $[p^{k-1}l, p^k l]$, and their widths belong to the same interval $[p^{k-1}w, p^k w]$. If l_1, l_2 are the lengths of R_1, R_2 and w_1, w_2 their widths, then $l_2 \leq pl_1$ and $w_2 \leq pw_1$. Let ϕ be the angle between R_1 and R_2. We shall prove that, if N is larger than a constant depending on C, then $R_1 \subseteq 2R_2$.

From the hypothesis, $R_2 \subset CR_1$, and hence $l_2 \sin\phi \leq w_1$. It follows that

$$\sin\phi \leq C^2 \tfrac{w}{l}.$$

Define $T_1 = \frac{1}{N}R_1$ and $T_2 = \frac{1}{N}R_2$. For $R_2 \subset 2R_1$, it suffices that $NT_2 \subset 2NT_1$. Without loss of generality, assume that the center of T_1 is the origin. Let $A = (A_x, A_y)$ be the center of T_2 and $B = (B_x, B_y)$ an arbitrary point of it. We want $A + N(B - A) \in 2NT_1$, and we see that it is enough that

i) $|A_x + N(B_x - A_x)| \leq l_1$,

ii) $|A_y + N(B_y - A_y)| \leq w_1$.

For i) we have

$$\begin{aligned} A_x &\leq \frac{\text{length } (T_1)}{2} + \frac{\text{diagonal } (T_2)}{2} = \frac{l_1 + \sqrt{w_2^2 + l_2^2}}{2N} \leq \frac{l_1 + \sqrt{2}l_2}{2N} \leq \frac{1 + \sqrt{2}p}{2N} l_1 \\ |B_x - A_x| &\leq \frac{\text{diagonal } (T_2)}{2} = \frac{\sqrt{w_2^2 + l_2^2}}{2N} \leq \frac{\sqrt{2}}{2}\frac{l_2}{N} \leq \frac{p}{\sqrt{2}N} l_1 \end{aligned}$$

Therefore, i) holds provided that

$$\frac{1 + \sqrt{2}C^{\frac{1}{N}}}{2N} + \frac{C^{\frac{1}{N}}}{\sqrt{2}} \leq 1,$$

which is true if N is larger than some constant depending only on C.

For part (ii), if θ is the angle between the diagonal and the long axis of R_2, then

$$\begin{aligned} |A_y| &\leq \frac{\text{width } (T_1)}{2} + \frac{\text{diagonal } (T_2)}{2}\sin(\theta + \phi) = \frac{w_1}{2N} + \frac{\sqrt{w_2^2 + l_2^2}}{2N}\sin(\theta + \phi) \\ &\leq \frac{w_1}{2N} + \frac{\sqrt{2}l_2}{2N}\sin\phi + \frac{\sqrt{2}l_2}{2N}\sin\theta \\ &\leq \frac{w_1}{2N} + \frac{w_2}{\sqrt{2}N} + \frac{\pi}{2\sqrt{2}}C^2\frac{l_2 w}{N^2 l} \\ &\leq \frac{w_1}{2N} + \frac{w_2}{N\sqrt{2}} + \frac{\pi}{2\sqrt{2}}C^3\frac{w}{N^2} = \left(\frac{1}{2N} + \frac{\pi C^3}{2\sqrt{2}N^2}\right) w_1 + \frac{w_2}{N\sqrt{2}} \\ &\leq \left(\frac{1}{2N} + \frac{\pi C^3}{2\sqrt{2}N^2}\right) w_1 + \frac{p w_1}{N\sqrt{2}} \end{aligned}$$

and similarly

$$|B_y - A_y| \leq \frac{\sqrt{w_2^2 + l_2^2}}{2N}\sin(\phi + \theta) \leq \frac{\pi}{2\sqrt{2}}\frac{C^3}{N^2}w_1 + \frac{p w_1}{N\sqrt{2}}$$

Finally, for ii) it is enough that

$$w_1\left(\frac{1}{2N} + \frac{\pi C^3}{2N^2\sqrt{2}}\right) + \frac{p w_1}{N\sqrt{2}} + \frac{\pi}{2\sqrt{2}}\frac{C^3}{N}w_1 + \frac{p w_1}{\sqrt{2}} \leq w_1$$

or, more simply,

$$\frac{1}{2N} + \frac{\pi C^3}{2\sqrt{2}N^2} + \frac{C^{\frac{1}{N}}}{N\sqrt{2}} + \frac{\pi C^3}{2N\sqrt{2}} + \frac{C^{\frac{1}{N}}}{\sqrt{2}} \leq 1,$$

which is possible if N is larger than a constant depending on C. $\square$

Proposition 2. *Let C be a constant and let ϵ, t satisfy $\epsilon < C$. If $\mathcal{F}$ is a family of rectangles with lengths between t and Ct and widths between ϵt and $2\epsilon t$, and if for every $T \in \mathcal{F}$ and every $\rho > 2\epsilon$, no rectangle with the same axis and center as T and width ρt contains more than $m\frac{\rho}{\epsilon}$ elements of $\mathcal{F}$, then*

$$\left\| \sum_{T\in\mathcal{F}} \chi_T \right\|_2^2 \leq Cm \log\tfrac{1}{\epsilon} \sum_{T\in\mathcal{F}} |T|.$$

Proof. Partition the interval $[0, \frac{\pi}{2}]$ into arcs $[0, \epsilon]$, $[\epsilon, 2\epsilon]$, … , $\left[2^{N-1}\epsilon, \frac{\pi}{2}\right]$ with $2^{N-1}\epsilon < \frac{\pi}{2} \leq 2^N\epsilon$, and fix a rectangle $T_i \in \mathcal{F}$. If some rectangle $T_j \in \mathcal{F}$ intersects T_i with axis angle ω, then their intersection forms a parallelogram of area

$$|T_i \cap T_j| = \frac{\text{width}\,(T_i)\ \text{width}\,(T_j)}{\sin\omega}.$$

Using the fact that the width is smaller than $2\epsilon t$ and that $\omega\frac{2}{\pi} \leq \sin\omega \leq \omega$,

$$|T_i \cap T_j| \leq C\frac{\epsilon^2 t^2}{\omega} \quad .$$

It follows that

$$\begin{aligned}
\left\| \sum_{T\in\mathcal{F}} \mathcal{X}_T \right\|_2^2 &= \int \sum_{T_i, T_j \in\mathcal{F}} \mathcal{X}_{T_i}\cdot\mathcal{X}_{T_j} = \sum_{T_i, T_j \in\mathcal{F}} \int \mathcal{X}_{T_i\cap T_j} = \sum_{T_i, T_j \in\mathcal{F}} |T_i \cap T_j| \\
&\leq \sum_{T_i\in\mathcal{F}} \left(\sum_{k=1}^{N} C\frac{\epsilon^2 t^2}{\omega} \#\left\{T_j :\ T_j \text{ intersects } T_i \text{ at angle } \omega : 2^{k-1}\epsilon \leq \omega \leq 2^k\epsilon\right\} \right. \\
&\qquad \left. + C\epsilon t^2 \#\left\{T_j : T_j \text{ intersects } T_i \text{ at angle } \omega : 0 \leq \omega \leq \epsilon\right\} \right)
\end{aligned}$$

Observe that $N \leq C\log\frac{1}{\epsilon}$. Finally, all the rectangles that intersect a fixed T_i at angle ω lie inside a rectangle with the same center and axis as T_i and width $\leq 2\epsilon t + 2Ct\omega = (2\epsilon + 2C\omega)t$, so by the hypothesis their number is at most $m\frac{2\epsilon+2C\omega}{\epsilon} \leq C\frac{m\omega}{\epsilon}$. Therefore

$$\begin{aligned}
\left\| \sum_{T\in\mathcal{F}} \mathcal{X}_T \right\|_2^2 &\leq C \sum_{T_i\in\mathcal{F}} \left(\sum_{k=1}^{C\log\frac{1}{\epsilon}} \frac{\epsilon^2 t^2}{\omega}\cdot\frac{m\omega}{\epsilon} + \epsilon t^2 m \right) \\
&\leq Cm\log\frac{1}{\epsilon} \sum_{T_i\in\mathcal{F}} \epsilon t^2 \leq Cm\log\frac{1}{\epsilon}\sum_{T\in\mathcal{F}} |T|\,.
\end{aligned}$$

□

Lemma 1. *Let C be a sufficiently large constant and Q a square in the plane with side length C. Let $\mathcal{F}$ be a family of rectangles of width $\delta = R^{-\frac{1}{2}}$, length 1, and cardinality $\#\mathcal{F} = \delta^{-100}$, all contained in Q. Suppose that whenever two rectangles in the family $\mathcal{F}$ intersect, they form an angle of at least δ. Then* *we can partition $\mathcal{F}$ into at most $C\left(\log\frac{1}{\delta}\right)^2$ subfamilies $\mathcal{F}_{ij}$ such that for each i and j there exist numbers $p = p(j)$ and $\theta = \theta(i) \geq \delta$ and a family of rectangles $\mathcal{G}_{ij} = \{\tau_k\}$ with length between 1 and C and width between θ and 2θ such that the following hold.*

1. *If $T \in \mathcal{F}_{ij}$, then there exists $\tau_k \in \mathcal{G}_{ij}$ with $T \subseteq \tau_k$.*
2. *If $T \in \mathcal{F}$, then T is contained in a bounded number of τ_k, $\tau_k \in \mathcal{G}_{ij}$.*
3. *Every $\tau_k \in \mathcal{G}_{ij}$ contains approximately $p\frac{\theta}{\delta}$ rectangles T from the family $\mathcal{F}$.*
4. $\| \sum_{\tau_k \in \mathcal{G}_{ij}} \chi_{\tau_k} \|_2^2 \leq C \log\frac{1}{\delta} \sum_{\tau_k \in \mathcal{G}_{ij}} |\tau_k|$.
5. $\| \sum_{T \in \mathcal{F}_{ij}} \chi_T \|_2^2 \leq Cp \log\frac{1}{\delta} \sum_{T \in \mathcal{F}_{ij}} |T|$.

Proof. For each rectangle $T \in \mathcal{F}$ we define Π_T to be the rectangle with the same center and sides parallel to T that maximizes the quantity

$$d(\Pi) = \delta\frac{\#\big(\{T' \in \mathcal{F} :\ T' \subseteq \Pi\}\big)}{|\Pi|}.$$

From the way Π_T is constructed, the following hold:

I. Π_T contains T, and at most it can contain all the elements of $\mathcal{F}$ that lie in the square Q of side length C. Hence $1 \leq$ length of $\Pi_T \leq 2\sqrt{2}C = C$ and $\delta \leq$ width of $\Pi_T \leq 2\sqrt{2}C = C$.

II. $\delta^{-1} \leq d(\Pi_T) \leq \delta\frac{\#\mathcal{F}}{|T|} = \delta^{-100}$.

We partition $\mathcal{F}$ as follows. If we denote the width of Π_T by θ_T, then for some $i \in \mathbb{Z}$ and some $j \in \mathbb{Z}$ we have $2^{-i-1} < \theta_T \leq 2^{-i}$ and $2^j < d(\Pi_T) \leq 2^{j+1}$. We define

$$\mathcal{F}_{ij} := \big\{T \in \mathcal{F} : \theta_T \in (2^{-i-1}, 2^{-i}] \text{ and } d(\Pi_T) \in (2^j, 2^{j+i}]\big\},$$

set $p = 2^j$ and $\theta = C_1 2^{-i-1}$, where C_1 is a constant to be determined later. We observe that the number of possible i satisfies $\#i \leq C\log\frac{1}{\delta}$ and similarly $\#j \leq C\log\frac{1}{\delta}$, hence $(\#i)(\#j) \leq C\left(\log\frac{1}{\delta}\right)^2$.

For each i, j we consider a maximal subset $\mathcal{G}^*_{ij}$ of $\mathcal{F}_{ij}$ such that if $T_1, T_2 \in \mathcal{G}^*_{ij}$, then $\Pi_{T_1} \nsubseteq 2\Pi_{T_2}$. We define $\mathcal{G}_{ij} := \{C_1\Pi_T : T \in \mathcal{G}^*_{ij}\}$.

To prove (1), let $T \in \mathcal{F}_{ij}$. If $C_1\Pi_T \in \mathcal{G}_{ij}$, then $T \subseteq C_1\Pi_T$ and we are done. If $C_1\Pi_T \notin \mathcal{G}_{ij}$, then there exists some T_1 with $C_1\Pi_{T_1} \in \mathcal{G}_{ij}$ and either $\Pi_T \subseteq 2\Pi_{T_1}$ or $\Pi_{T_1} \subseteq 2\Pi_T$. In the first case we are done. In the second case, observe that Π_{T_1}, Π_T have comparable widths and lengths, since their widths range from $C_1 2^{-i-1}$ to $C_1 2^{-i}$ while their lengths range from 1 to C. We conclude that, for a suitably large constant C_1, we obtain $T \subseteq \Pi_T \subseteq C_1\Pi_{T_1}$.

To prove (2), let $\tau_k \in \mathcal{G}_{ij}$ be two arbitrary rectangles that contain the same $T \in \mathcal{F}$. Since they have comparable dimensions, one is contained in a suitable dilation of the other. Taking into account that no τ_k is contained in twice another, Proposition 1 implies that the cardinality of such a set is bounded.

To prove (3), let $\tau_k = C_1\Pi_T \in \mathcal{G}_{ij}$ and let Π_T and T be the corresponding objects. Then, by definition,

$$d(\Pi_T) = \delta\frac{\#\{T' \in \mathcal{F} : T' \subseteq \Pi_T\}}{|\Pi_T|},$$

hence

$$p = 2^j \leq \delta\frac{\#\{T' \in \mathcal{F} : T' \subseteq \Pi_T\}}{\theta_T} \leq \delta C_1\frac{\#\{T' \in \mathcal{F} : T' \subseteq \Pi_T\}}{\theta}$$

and taking into account that $\Pi_T \subseteq \tau_k = C_1\Pi_T$ we find that

$$\#\{T' \in \mathcal{F} : T' \subseteq \tau_k\} \gtrsim p\frac{\theta}{\delta}.$$

On the other hand, Π_T maximizes the function d, hence

$$d(\tau_k) = \delta\frac{\#\{T' \in \mathcal{F} : T' \subseteq \tau_k\}}{|\tau_k|} \leq d\left(\Pi_T\right) \leq 2p.$$

Therefore

$$\#\{T' \in \mathcal{F} : T' \subseteq \tau_k\} \lesssim p\frac{|\tau_k|}{\delta} \lesssim p\frac{\theta}{\delta}.$$

Let $\tau_k = C_1\Pi_T \in \mathcal{G}_{ij}$ and let R be a rectangle with sides parallel to τ_k, the same center, bounded length, and width ρ. If R contains n rectangles $\tau_k \in \mathcal{G}_{ij}$, then by 2 and 3 it contains at least $Cnp\frac{\theta}{\delta}$ rectangles T from $\mathcal{F}$. By the maximality property of Π_T, we have $d(R) \leq d(\Pi_T)$, so

$$\delta\frac{np\frac{\theta}{\delta}}{\rho} \lesssim \delta\frac{p\frac{\theta}{\delta}}{\theta},$$

and hence

$$n \leq C\frac{\rho}{\theta}.$$

Applying Proposition 2 to the family $\mathcal{G}_{ij}$ with $m = C$, $t = 1$, $\epsilon = \theta$ establishes (4).

Let $T \in \mathcal{F}_{ij}$ and let Π be a rectangle with the same axis and center as T and width ρ. Suppose that Π contains n elements of $\mathcal{F}$. Since all the elements of $\mathcal{F}$ lie inside the square Q, we may assume that length$(\Pi) \leq C$. Then $d(\Pi) \leq d(\Pi_T)$, so $\delta\frac{n}{|\Pi|} \leq p$ and hence

$$n \leq p\frac{|\Pi|}{\delta} \leq \frac{Cp\rho}{\delta}.$$

Applying Proposition 2 with $m = Cp$, $t = 1$, $\epsilon = \delta$ establishes (5). $\square$

For each rectangle T in $\mathbb{R}^2$, let Γ_T denote the affine transformation that maps T onto the unit square. If, in addition, $\phi^{(k)}(x) = \min\{\|x\|^{-k}, 1\}$, where k is a natural number, then we define $\phi_T^{(k)} = \phi^{(k)} \circ \Gamma_T$.

With the notation of Lemma 1, we divide the unit circle into arcs β, each of length about $\delta = R^{-\frac{1}{2}}$, and also into arcs Θ of length about θ, so that each arc β is contained in some arc Θ. For each pair $\mathcal{F}_{ij}$, $\mathcal{G}_{ij}$ from Lemma 1 we consider the corresponding functions

$$\psi_\beta^{(k)} = \sum_{\substack{T\in\mathcal{F}_{ij},\\ \text{angle of } T\\ \text{lies in } \beta}} \phi_T^{(k)},$$

$$\psi_\Theta^{(k)} = \sum_{\substack{\tau\in\mathcal{G}_{ij},\\ \text{angle of } \tau\\ \text{lies in } \Theta}} \phi_\tau^{(k)}.$$

The functions $\psi_\beta^{(k)}$ and $\psi_\Theta^{(k)}$ correspond to a specific pair $\mathcal{F}_{ij}$, $\mathcal{G}_{ij}$.

Lemma 2. *Let $k \geq 2$ be a natural number and let $\mathcal{F}_{ij}$, $\mathcal{G}_{ij}$ be a pair. Then the functions $\psi_\beta^{(k)}$ and $\psi_\Theta^{(k)}$ satisfy*

$$\left\|\sum_\beta \psi_\beta^{(k)}\right\|_2^2 \leq C(k)p\log R \sum_{T\in\mathcal{F}_{ij}} |T|$$

and

$$\left\|\sum_\Theta \psi_\Theta^{(k)}\right\|_2^2 \leq C(k)\log R \sum_{\tau\in\mathcal{G}_{ij}} |\tau|.$$

Proof. For every $A \geq 1$ and every T in $\mathcal{F}_{ij}$ and τ in $\mathcal{G}_{ij}$, the rectangles AT and $A\tau$ have widths from $A\delta$ to $2A\delta$ and from $CA\theta$ to $2CA\theta$ respectively, and lengths from CA to $C'A$. Let $T \in \mathcal{F}_{ij}$. From the proof of 5 in Lemma 1 we know that every rectangle Π with the same axis and center as T and width ρ contains fewer than $\frac{Cp\rho}{\delta}$ rectangles T', where $T' \in \mathcal{F}$. By a simple geometric argument, it is clear that every rectangle Π with the same axis and center as AT and width $A\rho$ contains fewer than $\frac{CpA\rho}{\delta}$ rectangles AT', where $T' \in \mathcal{F}$. Applying Proposition 2 with $m = CpA$, $t = CA$, $\epsilon = \delta$, we find that

$$\left\|\sum_{T\in\mathcal{F}_{ij}} \chi_{AT}\right\|_2^2 \leq CAp\log\frac{1}{\delta}\sum_{T\in\mathcal{F}_{ij}} |AT| = CA^3p\log\frac{1}{\delta}\sum_{T\in\mathcal{F}_{ij}} |T|.$$

Similarly,

$$\left\|\sum_{\tau\in\mathcal{G}_{ij}} \chi_{A\tau}\right\|_2^2 \lesssim A^3\log R\sum_{T\in\mathcal{F}_{ij}} |T|.$$

We consider the auxiliary function

$$B_T = \sum_{m=0}^{+\infty} 2^{-km}\chi_{2^mT}.$$

We shall show that $\phi_T^{(k)} \lesssim B_T$. It suffices to show that $\phi_T^{(k)} \circ \Gamma_T^{-1} \lesssim B_T \circ \Gamma_T^{-1}$, where Γ_T^{-1} is the affine transformation that sends Q to T. In other words, we work with Q in place of T.

Let $x \in \mathbb{R}^2$. Then there exists $m \in \mathbb{Z}$ such that $2^{-km} \geq \phi^{(k)}(x) \geq 2^{-(m-1)k}$, and hence if $\|x\| \in [2^m, 2^{m+1}]$, then $\phi^{(k)}(x) \leq 2^{-km}\chi_{2^{m+1}Q}(x) = 2^k 2^{(-m-1)k}\chi_{2^{m+1}Q}(x)$. Finally, for all $x \in \mathbb{R}^2$,

$$\phi^{(k)}(x) \leq 2^k \sum_{m=0}^{+\infty} 2^{-km}\chi_{2^mQ}(x).$$

Therefore

$$\phi_T^{(k)} \leq 2^k B_T.$$

Hence we obtain

$$\begin{aligned}
\left\|\sum_{\beta}\psi_\beta^{(k)}\right\|_2^2 &= \left\|\sum_{\beta}\sum_{\substack{T\in\mathcal{F}_{ij},\\ \text{angle of } T\\ \text{lies in } \beta}}\phi_T^{(k)}\right\|_2^2 \\
&\leq 4^k\left\|\sum_{\beta}\sum_{\substack{T\in\mathcal{F}_{ij},\\ \text{angle of } T\\ \text{lies in } \beta}}\sum_{m=0}^{+\infty}2^{-km}\chi_{2^mT}\right\|_2^2 \\
&= 4^k\left\|\sum_{T\in\mathcal{F}_{ij}}\sum_{m=0}^{+\infty}2^{-km}\chi_{2^mT}\right\|_2^2 \\
&= 4^k\left\|\sum_{m=0}^{+\infty}\sum_{T\in\mathcal{F}_{ij}}2^{-km}\chi_{2^mT}\right\|_2^2 \\
&\overset{\text{Minkowski}}{\leq} 4^k\left(\sum_{m=0}^{+\infty}2^{-km}\left\|\sum_{T\in\mathcal{F}_{ij}}\chi_{2^mT}\right\|_2\right)^2 \\
&\leq 4^k\left(\sum_{m=0}^{+\infty}2^{-km}\left(2^{3m}p\log R\sum_{T\in\mathcal{F}_{ij}}|T|\right)^{\frac{1}{2}}\right)^2 \\
&= 4^k p\log R\sum_{T\in\mathcal{F}_{ij}}|T|\left(\sum_{m=0}^{+\infty}2^{-km}2^{\frac{3}{2}m}\right)^2 \\
&= C(k)p\log R\sum_{T\in\mathcal{F}_{ij}}|T|.
\end{aligned}$$

The second inequality is proved in the same way. □

Let A_R be the annulus $A_R = \{\xi : \; R-1 \leq \|\xi\| \leq R+1\}$, and partition it into disjoint annular sectors β of angular length approximately $\delta = \frac{1}{\sqrt{R}}$, that is, each β is of the form

$$\left\{\xi \; : \|\xi\| \in [R-1, R+1], \frac{\xi}{\|\xi\|} \in \gamma\right\},$$

where γ is an arc of the unit circle of length approximately $\frac{1}{\sqrt{R}}$. Next we enlarge each sector by a factor of C and denote the enlarged sector by $C\beta$; thus each $C\beta$ is of the form

$$C\beta = \left\{\xi \; : \|\xi\| \in [R-C, R+C], \frac{\xi}{\|\xi\|} \in \gamma'\right\},$$

where $\gamma' = C\gamma$ if $C\text{length}(\gamma) \leq 2\pi$, or $\gamma' = 2\pi$ otherwise. Let f be a function with the property that $f = \sum_\beta f_\beta$, where each f_β is supported in $C\beta$, and let $G_\beta = \widehat{f_\beta}$. Let Θ be another partition of the annulus A_R into annular sectors such that each β is contained in some Θ. We define

$$Sf = \left(\sum_\beta |G_\beta|^2\right)^{\frac{1}{2}}, \qquad S_\theta f = \left(\sum_\Theta \left|\sum_{\beta\subseteq\Theta} G_\beta\right|^2\right)^{\frac{1}{2}}.$$

Lemma 3. $\|S_\theta f\|_4 \leq C\|Sf\|_4$.

Proof. Suppose the inequality has already been proved when f is supported in an "approximate" octant of the circle. We then define

$$f_\beta^{(i)} = \begin{cases} f_\beta & \text{if } \beta \text{ is contained in the } i\text{th octant,} \\ 0 & \text{otherwise.} \end{cases}$$

We also define

$$f^{(i)} = \sum_\beta f_\beta^{(i)}.$$

Observe that $f_\beta = \sum_i f_\beta^{(i)}$ and $f = \sum_i f^{(i)}$. Then

$$\begin{aligned} S_\theta f = S_\theta\left(\sum_i f^{(i)}\right) &= \left(\sum_\Theta \left|\sum_{\beta\subseteq\Theta}\sum_i G_\beta^{(i)}\right|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_\Theta \left|\sum_i\sum_{\beta\subseteq\Theta} G_\beta^{(i)}\right|^2\right)^{\frac{1}{2}} \overset{\textit{Minkowski}}{\leq} \sum_i \left(\sum_\Theta \left|\sum_{\beta\subseteq\Theta} G_\beta^{(i)}\right|^2\right)^{\frac{1}{2}} \\ &= \sum_i S_\theta f^{(i)}. \end{aligned}$$

By the triangle inequality,

$$
\begin{aligned}
\|S_\theta f\|_4 &\le \sum_i \|S_\theta f^{(i)}\|_4 \le C \sum_i \|S f^{(i)}\|_4 \\
&= \sum_i \left(\int \left(\sum_\beta |G_\beta^{(i)}|^2 \right)^2 \right)^{\frac{1}{4}} \overset{\textit{Hölder}}{\lesssim} \left(\int \sum_i \left(\sum_\beta |G_\beta^{(i)}|^2 \right)^2 \right)^{\frac{1}{4}} \\
&\le \left(\int \left(\sum_i \sum_\beta |G_\beta^{(i)}|^2 \right)^2 \right)^{\frac{1}{4}} .
\end{aligned}
$$

Set $h_{ij} = \sum_\beta f_\beta^{(i)} * \widetilde{f_\beta^{(j)}}$ and $h = \sum_\beta f_\beta * \widetilde{f}_\beta$. Observe that $h_{ij} = 0$ if $i \neq j$. Then

$$
\sum_\beta |G_\beta^{(i)}|^2 = \sum_\beta \widehat{f_\beta^{(i)}} \overline{\widehat{f_\beta^{(i)}}} = \widehat{\left(\sum_\beta f_\beta^{(i)} * \widetilde{f_\beta^{(i)}} \right)} = \widehat{h_{ii}}.
$$

Also,

$$
\sum_\beta |G_\beta|^2 = \sum_\beta \widehat{f}_\beta \overline{\widehat{f}_\beta} = \widehat{\left(\sum_\beta f_\beta * \widetilde{f}_\beta \right)} = \widehat{h}.
$$

Therefore

$$
\begin{aligned}
\|S_\theta f\|_4 &= \left(\int \left| \sum_i \widehat{h_{ii}} \right|^2 \right)^{\frac{1}{4}} \overset{\text{Plancherel}}{=} \left(\int \left| \sum_i h_{ii} \right|^2 \right)^{\frac{1}{4}} \\
&= \left(\int \left| \sum_{i,j} h_{i,j} \right|^2 \right)^{\frac{1}{4}} = \left(\int |h|^2 \right)^{\frac{1}{4}} \overset{\text{Plancherel}}{=} \left(\int \left| \widehat{h} \right|^2 \right)^{\frac{1}{4}} .
\end{aligned}
$$

Hence, without loss of generality, we may assume that f is supported in the first octant.

$$
\begin{aligned}
\|S_\theta f\|_4^4 &= \left\| \sum_\Theta \left| \sum_{\beta \subseteq \Theta} G_\beta \right|^2 \right\|_2^2 \overset{\textit{Plancherel}}{=} \left\| \sum_\Theta \sum_{\beta, \beta_1 \subseteq \Theta} \widehat{G_\beta} * \widetilde{\widehat{G_{\beta_1}}} \right\|_2^2 \\
&= \sum_{\Theta_1, \Theta_2} \sum_{\beta, \beta_1 \subseteq \Theta_1} \sum_{\beta_2, \beta_3 \subseteq \Theta_2} \widehat{G_\beta} * \widetilde{\widehat{G_{\beta_1}}} * \widehat{G_{\beta_2}} * \widetilde{\widehat{G_{\beta_3}}}(0) \\
&= \sum_{\Theta_1, \Theta_2} \sum_{\beta, \beta_1 \subseteq \Theta_1} \sum_{\beta_2, \beta_3 \subseteq \Theta_2} \left(\widehat{G_\beta} * \widehat{G_{\beta_2}} \right) * \left(\widetilde{\widehat{G_{\beta_1}}} * \widetilde{\widehat{G_{\beta_3}}} \right) (0) \\
&= \sum_{\Theta_1, \Theta_2} \int \sum_{\substack{\beta \subseteq \Theta_1 \\ \beta_2 \subseteq \Theta_2}} \sum_{\substack{\beta_1 \subseteq \Theta_1 \\ \beta_3 \subseteq \Theta_2}} \left(\widehat{G_\beta} * \widehat{G_{\beta_2}} \right) \left(\overline{\widehat{G_{\beta_1}} * \widehat{G_{\beta_3}}} \right) = \sum_{\Theta_1, \Theta_2} \int \left| \sum_{\substack{\beta \subseteq \Theta_1 \\ \beta_2 \subseteq \Theta_2}} \widehat{G_\beta} * \widehat{G_{\beta_2}} \right|^2 .
\end{aligned}
$$

But $\widehat{G_\beta} = \widehat{\widehat{f_\beta}} = \widetilde{f_\beta}$ and each f_β is supported in $C\beta$, so each $f_{\beta_i} * f_{\beta_j}$ is supported in $C\beta_i + C\beta_j$. Every point $x \in \mathbb{R}^2$ belongs to only boundedly many such sets. If $k = k(C)$ denotes this number, then

$$\sum_{\Theta_1,\Theta_2} \int \Bigg| \sum_{\substack{\beta\subseteq\Theta_1\\ \beta_2\subseteq\Theta_2}} \widehat{G_\beta} * \widehat{G_{\beta_2}} \Bigg|^2 \overset{\mathit{Hölder}}{\leq} k \sum_{\Theta_1,\Theta_2} \sum_{\substack{\beta\subseteq\Theta_1\\ \beta_2\subseteq\Theta_2}} \int \left| \widehat{G_\beta} * \widehat{G_{\beta_2}} \right|^2$$

$$\overset{\mathit{Plancherel}}{=} k \sum_{\Theta_1,\Theta_2} \sum_{\substack{\beta\subseteq\Theta_1\\ \beta_2\subseteq\Theta_2}} \int |G_\beta G_{\beta_2}|^2 \ .$$

Finally,

$$\|S_\theta f\|_4^4 \lesssim \sum_{\Theta_1,\Theta_2} \sum_{\substack{\beta\subseteq\Theta_1\\ \beta_2\subseteq\Theta_2}} \int |G_\beta G_{\beta_2}|^2 = \|Sf\|_4^4.$$

□

The following lemma is a special form of the uncertainty principle.

Lemma 4. *Let $\phi = \phi^{(M)} = \min\{1, \|x\|^{-M}\}$ and let G be a function whose Fourier transform $\widehat{G}$ is supported in a rectangle ρ. Then for every rectangle σ, dual to ρ, one has*

$$\left\| \frac{G}{\phi_\sigma} \right\|_\infty^2 \leq C_M |\rho| \left\| \frac{G}{\phi_\sigma^2} \right\|_2^2,$$

where $\phi_\sigma = \phi \circ \Gamma_\sigma$ and Γ_σ is the affine transformation that sends σ to the unit square Q.

Proof. We first prove the claim for $\sigma = \rho = Q$. Let k be a Schwartz function such that $\widehat{k}(x) = 1$ whenever $x \in Q$. Then $\widehat{k * G} = \widehat{k}\,\widehat{G} = \widehat{G}$, because wherever $\widehat{G}$ is nonzero, $\widehat{k}$ equals 1. It follows that $k * G = G$ for almost every $x \in \mathbb{R}^2$. Then

$$\left\| \frac{G}{\phi} \right\|_\infty = \left\| \int \max\left\{1, \|x\|^M\right\} k(x-y) \min\left\{1, \|y\|^{-2M}\right\} \frac{G}{\phi^2}(y)\, dy \right\|_\infty$$

We observe that

$$\begin{aligned} \max\left\{1, \|x\|^M\right\} |k(x-y)| \min\left\{1, \|y\|^{-2M}\right\} &\leq C_M (1+\|x\|)^M (1+\|y\|)^{-2M} |k(x-y)| \\ &\leq C_M (1+\|x-y\|)^M (1+\|y\|)^{-M} |k(x-y)| \\ &\leq C_M (1+\|y\|)^{-M}. \end{aligned}$$

Therefore

$$\left\| \frac{G}{\phi} \right\|_\infty \leq C_M \int (1+\|y\|)^{-M} \left| \frac{G}{\phi^2}(y) \right| dy \leq C_M \left\| \frac{G}{\phi^2} \right\|_2 .$$

We now pass to the general case. Let ρ be a rectangle with center κ and let σ be a dual rectangle with center λ. If T is the linear transformation that sends Q to $\rho-\kappa$, then T^{-t} is the linear transformation that sends Q to $\sigma-\lambda$. Let $\widehat{G}$ be supported in ρ. Then

$$\left\|\frac{G}{\phi_\sigma}\right\|_\infty = |\det T| \left\|\frac{|\det T|^{-1}e^{-2\pi i\kappa T^{-t}\xi}G(T^{-t}\xi+\lambda)}{\phi_\sigma(T^{-t}\xi+\lambda)}\right\|_\infty .$$

But

$$\begin{aligned}\left(|\det T|^{-1}e^{-2\pi i\kappa T^{-t}\xi}G(T^{-t}\xi+\lambda)\right)^{\widehat{}} &= \int |\det T|^{-1}e^{-2\pi ix\cdot\xi}e^{-2\pi i\kappa T^{-t}\xi}G(T^{-t}\xi+\lambda)\,d\xi\\ &= \int |detT|^{-1}e^{-2\pi i\left(xT^tT^{-t}\xi+\kappa T^{-t}\xi\right)}G(T^{-t}\xi+\lambda)\,d\xi\\ &= \int |\det T|^{-1}e^{-2\pi i(Tx+\kappa)T^{-t}\xi}G(T^{-t}\xi+\lambda)\,d\xi\\ &= \int e^{-2\pi i(Tx+\kappa)\xi}G(\xi+\lambda)\,d\xi = e^{2\pi i(Tx+\kappa)\lambda}\widehat{G}(Tx+\kappa).\end{aligned}$$

Since $\phi_\sigma(T^{-t}x+\lambda)=\phi(x)$ and $|\det T|^{-1}e^{-2\pi i\kappa T^{-t}x}G(T^{-t}x+\lambda)$ has Fourier transform supported in Q, the special case implies

$$\begin{aligned}\left\|\frac{G}{\phi_\sigma}\right\|_\infty =& |\det T| \left\|\frac{|\det T|^{-1}e^{-2\pi i\kappa T^{-t}x}G(T^{-t}x+\lambda)}{\phi_\sigma(T^{-t}x+\lambda)}\right\|_\infty\\ & C_M|\rho| \left\|\frac{|\det T|^{-1}e^{-2\pi i\kappa T^{-t}x}G(T^{-t}x+\lambda)}{{\phi_\sigma}^2(T^{-t}x+\lambda)}\right\|_2\\ & \le C_M|\rho|^{\frac{1}{2}}\left\|\frac{G}{{\phi_\sigma}^2}\right\|_2 .\end{aligned}$$

□

Lemma 5. *Let μ be a measure supported in the unit disk and with α-dimensional energy $I_\alpha(\mu)=1$. Then for every $R\ge 2$ we can decompose μ into a sum of at most $C\log R$ measures μ_j such that*

$$\mu_j\left(\mathbb{R}^2\right)\sup_x \sup_{r\ge R^{-1}}\frac{\mu_j\big(D(x,r)\big)}{r^\alpha}\lesssim 1.$$

Proof. For every $x\in D(0;1)$ we have $D(0;1)\subseteq D(x;2)$, and hence

$$\sup_{r\ge R^{-1}}\frac{\mu\big(D(x;r)\big)}{r^\alpha}\ge\frac{\mu\big(D(x;2)\big)}{2^\alpha}\ge\frac{\mu\ \big(\ D(0;1)\ \big)}{2^\alpha}=\frac{1}{2^\alpha}\mu\left(\mathbb{R}^2\right).$$

On the other hand,

$$\sup_{r\ge R^{-1}}\frac{\mu\big(D(x;r)\big)}{r^\alpha}\le\frac{\mu\big(\mathbb{R}^2\big)}{R^{-\alpha}}=\mu\big(\mathbb{R}^2\big)R^\alpha.$$

We consider the sets

$$E_j=\left\{x:\quad\sup_{r\ge R^{-1}}\frac{\mu\big(D(x,r)\big)}{r^\alpha}\in\left[2^{j-1}\mu\big(\mathbb{R}^2\big),2^j\mu\big(\mathbb{R}^2\big)\right)\right\}.$$

We define μ_j to be the restriction of μ to E_j and observe that the number of j for which E_j is nonempty is $C \log R$. One has

$$\sup_{x \in \mathbb{R}^2} \sup_{r \geq R^{-1}} \frac{\mu_j\big(D(x,r)\big)}{r^\alpha} \lesssim 2^j \mu\left(\mathbb{R}^2\right).$$

Indeed, if $x \in E_j$, it is obvious that

$$\sup_{r \geq R^{-1}} \frac{\mu_j\big(D(x,r)\big)}{r^\alpha} \leq 2^j \mu\left(\mathbb{R}^2\right).$$

If $x \notin E_j$, then we may assume that $D(x,r) \cap E_j \neq \emptyset$. For $y \in D(x,r) \cap E_j$ we have $D(x,r) \subseteq D(y,2r)$, so for $r \geq R^{-1}$

$$\frac{\mu_j\left(D(x,r)\right)}{r^\alpha} \leq 2^\alpha \frac{\mu_j\left(D(y,2r)\right)}{(2r)^\alpha} \leq 2^\alpha \sup_{y \in E_j} \sup_{r \geq R^{-1}} \frac{\mu_j\left(D(y,r)\right)}{r^\alpha} \leq 2^\alpha 2^j \mu\left(\mathbb{R}^2\right),$$

and the desired conclusion follows for every $x \in \mathbb{R}^2$. Therefore, to prove the lemma it remains to show that for every j,

$$\mu_j\left(\mathbb{R}^2\right) = \mu(E_j) \lesssim \left(2^j \mu\left(\mathbb{R}^2\right)\right)^{-1}.$$

For every j, the set E_j is a bounded subset of $\mathbb{R}^2$, and each of its points is the center of some disk $D(x,r)$ such that $\mu\big(D(x,r)\big) \geq 2^{j-1}\mu\big(\mathbb{R}^2\big)r^\alpha$. By the Besicovitch covering lemma, there exists an at most countable family of such disks, $D_n = D_n(x,r)$, covering all of E_j, and each point of $\mathbb{R}^2$ belongs to at most C of these disks. For each n we have

$$\int_{D_n \times D_n} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} \geq \int_{D_n \times D_n} \frac{d\mu(x)d\mu(y)}{(2r)^\alpha} = \frac{1}{(2r)^\alpha} \cdot \mu(D_n) \cdot \mu(D_n) \gtrsim \mu(D_n) 2^j \mu\big(\mathbb{R}^2\big).$$

Finally, we have

$$\begin{aligned} 2^j \mu\left(\mathbb{R}^2\right) \mu\left(E_j\right) \leq 2^j \mu\left(\mathbb{R}^2\right) \sum_n \mu(D_n) &\lesssim \sum_n \int_{D_n \times D_n} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} \\ &= \int_{\mathbb{R}^2} \sum_n \chi_{D_n \times D_n} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} \\ &\leq C \int_{\mathbb{R}^2} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} = C I_\alpha(\mu). \end{aligned}$$

□

Lemma 6. *Suppose that μ is a measure supported in the unit disk and that*

$$\frac{1}{R} \int_{A_R} \left|\widehat{\mu}\,(x)\right|^2 \, dx \lesssim B \mu\left(\mathbb{R}^2\right) R^{-\frac{\alpha}{2}+\epsilon}$$

for every $R \geq 1$. Then

$$\int_{-\pi}^{\pi} \left|\widehat{\mu}\big(Re^{i\theta}\big)\right|^2 \, d\theta \leq C B \mu\left(\mathbb{R}^2\right) R^{-\frac{\alpha}{2}+\epsilon}.$$

Proof. Let k be a Schwartz function equal to one on the support of μ (the unit disk). Since $\widehat{\mu}=\widehat{k}*\widehat{\mu}$, we have

$$\begin{aligned}
\int_{-\pi}^{\pi}\left|\widehat{\mu}\left(Re^{i\theta}\right)\right|^{2}\,d\theta &= \int_{-\pi}^{\pi}\left|\widehat{k}*\widehat{\mu}\left(Re^{i\theta}\right)\right|^{2}\,d\theta\\
&\leq \int_{-\pi}^{\pi}\left(\int_{\mathbb{R}^2}\left|\widehat{k}\left(Re^{i\theta}-y\right)\widehat{\mu}(y)\right|\,dy\right)^{2}\,d\theta\\
&= \int_{-\pi}^{\pi}\left(\int_{\mathbb{R}^2}\left|\widehat{k}\left(Re^{i\theta}-y\right)\right|^{\frac{1}{2}}|\widehat{\mu}(y)|\left|\widehat{k}\left(Re^{i\theta}-y\right)\right|^{\frac{1}{2}}\,dy\right)^{2}\,d\theta\\
&\overset{\text{Hölder}}{\leq} \int_{-\pi}^{\pi}\int_{\mathbb{R}^2}|\widehat{\mu}(y)|^{2}\left|\widehat{k}\left(Re^{i\theta}-y\right)\right|\,dy\int_{\mathbb{R}^2}\left|\widehat{k}\left(Re^{i\theta}-y\right)\right|\,dy\,d\theta\\
&\leq C\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}|\widehat{\mu}(y)|^{2}\left|\widehat{k}\left(Re^{i\theta}-y\right)\right|\,d\theta\,dy.
\end{aligned}$$

Since k is Schwartz,

$$\begin{aligned}
\int_{-\pi}^{\pi}\left|\widehat{k}\left(Re^{i\theta}-y\right)\right|\,d\theta &\leq C\int_{-\pi}^{\pi}\frac{d\theta}{\left(1+|Re^{i\theta}-y|\right)^{102}}\\
&\leq \frac{C}{R}\int_{-\pi}^{\pi}\frac{Rd\theta}{\left(1+|Re^{i\theta}-y|\right)^{2}}\cdot\frac{1}{\left(1+|R-|y||\right)^{100}}\\
&\leq CR^{-1}\left(1+|R-|y||\right)^{-100}.
\end{aligned}$$

Therefore

$$\int_{-\pi}^{\pi}\left|\widehat{\mu}\left(Re^{i\theta}\right)\right|^{2}\,d\theta\leq C\int_{\mathbb{R}^2}R^{-1}(1+|R-|y||)^{-100}|\widehat{\mu}(y)|^{2}\,dy.$$

From here on, we have

$$\begin{aligned}
\int_{\mathbb{R}^2}R^{-1}(1+|R-|y||)^{-100}|\widehat{\mu}(y)|^{2}\,dy &\leq \frac{1}{R}\sum_{m=\left[-\frac{R}{2}\right]}^{+\infty}\int\limits_{R+2m-1\leq|y|\leq R+2m+1}\frac{1}{(1+|R-|y||)^{100}}|\widehat{\mu}(y)|^{2}\,dy\\
&\leq \frac{1}{R}\sum_{m=\left[-\frac{R}{2}\right]}^{+\infty}\frac{1}{(1+|m|)^{100}}\int\limits_{R+2m-1\leq|y|\leq R+2m+1}|\widehat{\mu}(y)|^{2}\,dy\\
&\leq B\mu\left(\mathbb{R}^2\right)\frac{1}{R}\sum_{m=\left[-\frac{R}{2}\right]}^{+\infty}\frac{R+2m}{(1+|m|)^{100}}(R+2m)^{-\frac{\alpha}{2}+\epsilon}.
\end{aligned}$$

Moreover,

$$
\begin{aligned}
\sum_{m=\left[-\frac{R}{2}\right]}^{+\infty} \frac{(R+2m)^{1-\frac{\alpha}{2}+\epsilon}}{(1+|m|)^{100}} &\leq C \sum_{m=\left[-\frac{R}{2}\right]}^{\left[\frac{R}{2}\right]} \frac{R^{1-\frac{\alpha}{2}+\epsilon}}{(1+|m|)^{100}} + \sum_{m=\left[\frac{R}{2}\right]}^{+\infty} \frac{m^{1-\frac{\alpha}{2}+\epsilon}}{m^{100}} \\
&= CR^{1-\frac{\alpha}{2}+\epsilon} \sum_{m=0}^{\left[\frac{R}{2}\right]} \frac{1}{(1+m)^{100}} + \sum_{m=\left[\frac{R}{2}\right]}^{+\infty} \frac{1}{m^{99+\frac{\alpha}{2}-\epsilon}} \\
&\leq CR^{1-\frac{\alpha}{2}+\epsilon} + C\frac{1}{R^{98+\frac{\alpha}{2}-\epsilon}} \\
&\leq CR^{1-\frac{\alpha}{2}+\epsilon}.
\end{aligned}
$$

Finally,

$$
\int_{-\pi}^{\pi} \left|\widehat{\mu}\left(Re^{i\theta}\right)\right|^2 \, d\theta \leq B\mu\left(\mathbb{R}^2\right) R^{-\frac{\alpha}{2}+\epsilon}.
$$

$\square$

Fix a function f supported in the annulus $A_R = \{\xi : \ R-1 \leq |\xi| \leq R+1\}$ with $\|f\|_{L^2} = 1$, and set $G = \widehat{f(-x)}$. Let h be a positive Schwartz function, and define

$$
H(x) = \frac{h(x)}{\sum_{k\in\mathbb{Z}^2} h(x+k)},
$$

for which

$$
\sum_{m\in\mathbb{Z}^2} H(x+m) = \sum_{m\in\mathbb{Z}^2} \frac{h(x+m)}{\sum_{k\in\mathbb{Z}^2} h(x+k+m)} = \frac{\sum_{m\in\mathbb{Z}^2} h(x+m)}{\sum_{k\in\mathbb{Z}^2} h(x+k)} = 1.
$$

We also consider a positive Schwartz function g with $\widehat{g}$ supported in Q and $\int g = 1$, and define $b = H * g$. Then $\widehat{b} = \widehat{H}\widehat{g}$ is supported in Q, and

$$
\sum_{m\in\mathbb{Z}^2} b(x+m) = \sum_{m\in\mathbb{Z}^2} \int H(x+m-y)g(y)\,dy = \int \sum_{m\in\mathbb{Z}^2} H(x+m-y)g(y)\,dy = \int g(y)\,dy = 1.
$$

Therefore, if we tile $\mathbb{R}^2$ by rectangles ρ^* dual to a given rectangle ρ, then $\sum_{\rho^*} b_{\rho^*} = 1$. Next we partition the annulus A_R into pairwise disjoint circular rectangles β of angular length approximately $R^{-\frac{1}{2}}$, each of which is contained in a standard rectangle ρ_β with dimensions C and $CR^{\frac{1}{2}}$. We write $G = \sum_\rho G_\rho$ with $G_\rho = \widehat{\left(\chi_\beta f\right)(-x)}$. For each $\rho = \rho_\beta$ and for all ρ^* dual to ρ and tiling $\mathbb{R}^2$, we define $G_\rho^{\rho^*} = b_{\rho^*}G_\rho$. Moreover, for each ρ, the function $\widehat{b_{\rho^*}}$ is supported in the ρ corresponding to the given β. Combined with Lemma 4, this gives

$$
\begin{aligned}
\sum_\rho \sum_{\rho^*} \left\| \frac{G_\rho^{\rho^*}}{\phi_{\rho^*}} \right\|_\infty^2 &= \sum_\rho \sum_{\rho^*} \left\| \frac{G_\rho b_{\rho^*}}{\phi_{\rho^*}} \right\|_\infty^2 \\
&\lesssim \sum_\rho \sum_{\rho^*} |\rho| \left\| \frac{G_\rho b_{\rho^*}}{\phi_{\rho^*}^2} \right\|_2^2 \lesssim R^{\frac{1}{2}} \sum_\rho \sum_{\rho^*} \left\| \frac{G_\rho b_{\rho^*}}{\phi_{\rho^*}^2} \right\|_2^2.
\end{aligned}
$$

Since b is Schwartz and $\frac{1}{\phi^2}$ has polynomial growth $\|x\|^{2M}$, we have $\left|\frac{b}{\phi^2}\right| \leq C_M$. Therefore, for each ρ^*,

$$\left|\frac{b_{\rho^*}}{\phi_\rho^{*2}}\right| \leq C_M.$$

Hence

$$\sum_{\rho^*}\left\|\frac{b_{\rho^*}}{\phi^2_{\rho^*}}G_\rho\right\|_2^2 = \int_{\mathbb{R}^2}\sum_{\rho^*}\left|\frac{b_{\rho^*}}{\phi^2_{\rho^*}}G_\rho\right|^2 = \int_{\mathbb{R}^2}|G_\rho|^2\sum_{\rho^*}\left|\frac{b_{\rho^*}}{\phi^2_{\rho^*}}\right|^2 \leq C_M\int_{\mathbb{R}^2}|G_\rho|^2\sum_{\rho^*}b_{\rho^*} = C_M\|G_\rho\|_2^2.$$

Finally,

$$\sum_\rho\sum_{\rho^*}\left\|\frac{G_\rho^{\rho^*}}{\phi_{\rho^*}}\right\|_\infty^2 \leq C_M R^{\frac{1}{2}}\sum_\rho\|G_\rho\|_2^2 = C_M R^{\frac{1}{2}}\sum_\rho\|\widehat{G_\rho}\|_2^2 = C_M R^{\frac{1}{2}}\|\chi_\beta f\|_2^2 \leq C_M R^{\frac{1}{2}}.$$

Summarizing, we have shown that

$$\sum_\rho\sum_{\rho^*}\left\|\frac{G_\rho^{\rho^*}}{\phi_{\rho^*}}\right\|_\infty^2 \leq C_M R^{\frac{1}{2}}. \tag{1}$$

From the hypothesis $B = \sup_x \sup_{r\geq R^{-1}}\frac{\mu(D(x;r))}{r^\alpha}$ it follows immediately that $\mu\left(\mathbb{R}^2\right) \leq CB$. If

$$\left|\int G\,d\mu\right| \leq \mu\left(\mathbb{R}^2\right)R^{-10},$$

then for every f supported in A_R with $\|f\|_{L^2} = 1$ it follows that

$$\left|\int_{A_R}\widehat{\mu}(x)f(-x)\,dx\right| = \left|\int G\,d\mu\right| \leq \mu\left(\mathbb{R}^2\right)R^{-10}.$$

Therefore

$$\int_{A_R}|\widehat{\mu}(x)|^2\,dx \leq \mu\left(\mathbb{R}^2\right)^2R^{-20} \leq B\mu\left(\mathbb{R}^2\right)R^{-20}$$

and

$$\frac{1}{R}\int_{A_R}|\widehat{\mu}(x)|^2\,dx \leq B\mu\left(\mathbb{R}^2\right)R^{-\frac{\alpha}{2}+\epsilon}.$$

Hence we may assume that

$$\mu\left(\mathbb{R}^2\right)R^{-10} \leq \left|\int G\,d\mu\right|.$$

Moreover, from (1) we see that

$$G = \sum_\rho\sum_{\rho^*}G_\rho^{\rho^*} = \sum_{\rho,\rho^*\colon\|\phi_{\rho^*}^{-1}G_\rho^{\rho^*}\|_\infty\leq C_M R^{\frac{1}{4}}}G_\rho^{\rho^*}.$$

Now

$$\left|\int \sum_{\rho,\rho^*:\ \left\|G_\rho^{\rho^*}\phi_{\rho^*}^{-1}\right\|_\infty \le R^{-100}} G_\rho^{\rho^*}\, d\mu\right|$$

$$= \left|\int \sum_{\rho,\rho^*:\ \left\|G_\rho^{\rho^*}\phi_{\rho^*}^{-1}\right\|_\infty \le R^{-100}} \phi_{\rho^*}\phi_{\rho^*}^{-1}G_\rho^{\rho^*}\, d\mu\right|$$

$$\le \|\phi_{\rho^*}^{-1}G_\rho^{\rho^*}\|_\infty \int \sum_{\rho,\rho^*:\ \left\|G_\rho^{\rho^*}\phi_{\rho^*}^{-1}\right\|_\infty \le R^{-100}} \phi_{\rho^*}\, d\mu \le \mu\left(\mathbb{R}^2\right) R^{-50}.$$

Therefore

$$\left|\int \sum_{\rho,\rho^*:\ R^{-100} \le \|\phi_{\rho^*}^{-1}G_\rho^{\rho^*}\|_\infty \le C_M R^{\frac{1}{4}}} G_\rho^{\rho^*}\, d\mu\right| \ge$$

$$\left|\int G d\mu\right| - \left|\int \sum_{\rho,\rho^*:\ \|\phi_{\rho^*}^{-1}G_\rho^{\rho^*}\|_\infty \le R^{-100}} G_\rho^{\rho^*}\, d\mu\right|$$

$$\ge \left|\int G d\mu\right| - \mu\left(\mathbb{R}^2\right) R^{-50} \ge \left|\int G d\mu\right| - \left|\int G d\mu\right| R^{-40} \ge C\left|\int G d\mu\right|.$$

In the usual way, we divide the ρ^* into $N \le C\log R$ families by partitioning $[R^{-100}, CR^{\frac{1}{4}}]$ into dyadic intervals:

$$\mathcal{F}_i = \{\rho^* : 2^{i-1}R^{-100} \le \|\phi_{\rho^*}^{-1}G_\rho^{\rho^*}\|_\infty \le 2^i R^{-100}\}.$$

Therefore,

$$\left|\sum_{i=1}^N \int \sum_{\rho,\rho^*\in\mathcal{F}_i} G_{\rho^*}^{\rho}\right| = \left|\int \sum_{\rho,\rho^*:\ R^{-100} \le \|\phi_{\rho^*}^{-1}G_\rho^{\rho^*}\|_\infty \le C_M R^{\frac{1}{4}}} G_\rho^{\rho^*}\, d\mu\right| \ge C\left|\int G d\mu\right|.$$

Hence there exists an i such that

$$\left|\int \sum_{\rho,\rho^*:\rho^*\in\mathcal{F}_i} G_\rho^{\rho^*} d\mu\right| \ge C\frac{1}{N}\left|\int G d\mu\right| \ge \frac{C}{\log R}\left|\int G d\mu\right|.$$

Fix $\epsilon > 0$. We now restrict the ρ^* even further by replacing $\mathcal{F}_i$ with a subfamily $\mathcal{F}_1$ (depending on ϵ), all of whose elements lie in a fixed square of side length 10, at the cost that $\frac{1}{\log R}$ becomes $R^{-\frac{\epsilon}{2}}$. Indeed, if R is sufficiently large, then this square lies inside the disk centered at the origin and of radius $R^{\frac{\epsilon}{5}}$, and hence

$$\left|\int \sum_{\rho,\rho^*:\rho^*\in\mathcal{F}_i} G_\rho^{\rho^*} d\mu\right| \le \left|\int \sum_{\substack{\rho,\rho^*:\rho^*\in\mathcal{F}_i,\\ d(\rho^*,0)>R^{\frac{\epsilon}{5}}}} G_\rho^{\rho^*} d\mu\right| + \left|\int \sum_{\substack{\rho,\rho^*:\rho^*\in\mathcal{F}_i,\\ d(\rho^*,0)\le R^{\frac{\epsilon}{5}}}} G_\rho^{\rho^*} d\mu\right|.$$

Moreover,

$$
\begin{aligned}
\left|\int \sum_{\substack{\rho,\rho^*:\rho^*\in\mathcal{F}_i,\\ d(\rho^*,0)>R^{\frac{\epsilon}{5}}}} G_\rho^{\rho^*}\,d\mu\right| &= \left|\int \sum_{\substack{\rho,\rho^*:\rho^*\in\mathcal{F}_i,\\ d(\rho^*,0)>R^{\frac{\epsilon}{5}}}} G_\rho^{\rho^*}\phi_{\rho^*}^{-1}\phi_{\rho^*}\,d\mu\right|\\
&\le \|G_\rho^{\rho^*}\phi_{\rho^*}^{-1}\|_\infty \int \sum_{\substack{\rho,\rho^*:\rho^*\in\mathcal{F}_i,\\ d(\rho^*,0)>R^{\frac{\epsilon}{5}}}} \phi_{\rho^*}\,d\mu\\
&\le 2^i R^{-100}\int \sum_{\substack{\rho,\rho^*:\rho^*\in\mathcal{F}_i\\ d(\rho^*,0)>R^{\frac{\epsilon}{5}}}} \phi_{\rho^*}\,d\mu\\
&\le CR^{\frac{1}{4}} \sum_{\substack{\rho,\rho^*:\rho^*\in\mathcal{F}_i,\\ d(\rho^*,0)>R^{\frac{\epsilon}{5}}}} \frac{1}{(d(\rho^*,0))^M}\mu\left(\mathbb{R}^2\right)\\
&\lesssim R^{\frac{1}{4}}\mu\left(\mathbb{R}^2\right)\int_{|x|>R^{\frac{\epsilon}{5}}}\frac{1}{|x|^M}\,dx\\
&\lesssim \frac{R^{\frac{1}{4}}}{R^{\frac{\epsilon}{5}(M-2)}}\mu\left(\mathbb{R}^2\right) \lesssim \mu\left(\mathbb{R}^2\right)R^{-100} \lesssim R^{-90}\left|\int G\,d\mu\right|.
\end{aligned}
$$

For the penultimate inequality to hold, $M = M(\epsilon)$ must be sufficiently large. Now the disk centered at 0 and of radius $R^{\frac{\epsilon}{5}}$ can be covered by at most $N' \le CR^{\frac{\epsilon}{5}}$ squares Q_λ of side length 10, with small overlap, so that each $\rho^* \in \mathcal{F}_i$ is contained in some Q_λ. Then

$$
\sum_{\lambda=1}^{N'}\left|\int \sum_{\rho,\rho^*:\rho^*\subseteq Q_\lambda} G_\rho^{\rho^*}\,d\mu\right| \ge \left(\frac{C}{\log R} - R^{-90}\right)\left|\int G d\mu\right|.
$$

Hence, for some Q_λ,

$$
\left|\int \sum_{\rho,\rho^*:\rho^*\subseteq Q_\lambda} G_\rho^{\rho^*}\,d\mu\right| \ge CR^{-\frac{\epsilon}{5}}\left|\frac{C}{\log R} - R^{-90}\right|\left|\int G d\mu\right| \ge CR^{-\frac{\epsilon}{2}}\left|\int G\,d\mu\right|.
$$

We define $\mathcal{F}_1$ to be the family of those ρ^* that belong to $\mathcal{F}_i$ and are contained in Q_λ.

The family of rectangles $\mathcal{F}_1$ has the following properties.

- If two elements ρ_1^*, ρ_2^* are dual to the same original ρ, then they intersect along their boundaries, since they form a tiling of $\mathbb{R}^2$. In the opposite case, their angle is equal to the angle of the corresponding ρ, and consequently is greater than $CR^{-\frac{1}{2}} = C\delta$.
- Since the area of each ρ^* is $CR^{-\frac{1}{2}}$, for each ρ there are $CR^{\frac{1}{2}}$ rectangles $\rho^* \subseteq Q_\lambda$. Therefore $\#\mathcal{F}_1 \le CR^{\frac{1}{2}}R^{\frac{1}{2}} \le CR^{50} = C\delta^{-100} = K$.
- The length of the elements of $\mathcal{F}_1$ is C and the width is $C\delta$.

- We have $\frac{h}{2} \leq \|\phi_{\rho^*}{}^{-1} G_\rho^{\rho^*}\|_\infty \leq h$, for every $\rho^* \in \mathcal{F}_1$, where $h = 2^i R^{-100}$ and $R^{-100} \leq h \leq CR^{\frac{1}{4}}$.
- From (1) it follows that
$$\sum_{\rho,\rho^*:\rho^* \in \mathcal{F}_1} \left\| \frac{G_\rho^{\rho^*}}{\phi_{\rho^*}} \right\|_\infty^2 \leq C_M R^{\frac{1}{2}},$$
and from $\frac{h}{2} \leq \left\|\phi_{\rho^*}^{-1} G_\rho^{\rho^*}\right\|_\infty$ it follows that $h^2 \# (\mathcal{F}_1) \leq C_M R^{\frac{1}{2}}$.

By Lemma 1 we can split $\mathcal{F}_1$ into at most $C \left(\log \frac{1}{\delta}\right)^2$ subfamilies $\mathcal{F}_{ij}$ for which the following hold:

- There exist numbers p and $\theta \geq R^{-\frac{1}{2}}$ and a set $\mathcal{G}_{ij}$ of rectangles τ with length C and width approximately θ, such that each $\rho^* \in \mathcal{F}_{ij}$ is contained in at least one $\tau \in \mathcal{G}_{ij}$.
- Each $\tau \in \mathcal{G}_{ij}$ contains approximately $p\theta\sqrt{R}$ rectangles from $\mathcal{F}_1$.
- Each $\rho^* \in \mathcal{F}_1$ is contained in at most C members of $\mathcal{F}_{ij}$.
- $\# (\mathcal{G}_{ij}) \lesssim \frac{\#\mathcal{F}_1}{p\theta\sqrt{R}} \leq \frac{K}{p\theta\sqrt{R}}$.

Also, the following hold:

- $$\left| \int \sum_{\rho,\rho^*:\rho^* \in \mathcal{F}_1} G_\rho^{\rho^*} \, d\mu \right| \geq CR^{-\frac{\epsilon}{2}} \left| \int G \, d\mu \right|.$$
- If $\rho^* \in \mathcal{F}_1$, then $\left|G_\rho^{\rho^*}\right| \leq h\phi_{\rho^*}$. This follows from the fact that $\|\phi_{\rho^*}^{-1} G_\rho^{\rho^*}\|_\infty \leq h$.

From (2) we have
$$\left| \int \sum_{i,j} \sum_{\rho,\rho^*:\rho^* \in \mathcal{F}_{ij}} G_\rho^{\rho^*} \, d\mu \right| \geq CR^{-\frac{\epsilon}{2}} \left| \int G \, d\mu \right|.$$

Therefore there exists a pair (i, j) such that
$$\left| \int \sum_{\rho,\rho^*:\rho^* \in \mathcal{F}_{ij}} G_\rho^{\rho^*} \, d\mu \right| \geq \frac{CR^{-\frac{\epsilon}{2}} \left|\int G \, d\mu\right|}{\#(i,j)} \geq \frac{CR^{-\frac{\epsilon}{2}} \left|\int G \, d\mu\right|}{(\log R)^2} \geq CR^{-\epsilon} \left| \int G \, d\mu \right|. \tag{2}$$

We set $\mathcal{F} = \mathcal{F}_{ij}$ and $\mathcal{G} = \mathcal{G}_{ij}$.

We summarize the properties of $\mathcal{F}$ and $\mathcal{G}$:

- $\#(\mathcal{G}) \leq C\frac{K}{p\theta\sqrt{R}}$.
- Each $\tau \in \mathcal{G}$ contains approximately $p\theta\sqrt{R}$ rectangles from $\mathcal{F}_1$.
- $\theta \geq R^{-\frac{1}{2}}$.
- Each $\rho^* \in \mathcal{F}$ is contained in at least one $\tau \in \mathcal{G}$.
- $\left| \int \sum_{\rho,\rho^*:\rho^* \in \mathcal{F}} G_\rho^{\rho^*} d\mu \right| \geq CR^{-\epsilon} \left|\int G \, d\mu\right|$.
- If $\rho^* \in \mathcal{F}$, then $\left|G_\rho^{\rho^*}\right| \leq h\phi_{\rho^*}$.
- $h^2 \# (\mathcal{F}) \leq C_M R^{\frac{1}{2}}$.

Now divide the unit circle into arcs Θ of length approximately θ so that the radial projection of each β is contained in some Θ. For each ρ let $\mathcal{F}(\rho)$ denote the family of all $\rho^* \in \mathcal{F}$ dual to ρ, and let $\mathcal{G}(\Theta)$ denote all $\tau \in \mathcal{G}$ whose angle lies in $C_1\Theta$, where C_1 is a constant to be specified shortly. Applying Lemma 2, define

$$\psi_\rho^{(k)} = \sum_{\rho^* \in \mathcal{F}(\rho)} \phi_{\rho^*}^{(k)}, \qquad \Psi_\Theta^{(k)} = \sum_{\tau \in \mathcal{G}(\Theta)} \phi_\tau^{(k)}.$$

Then Lemma 2 gives

$$\begin{aligned} \left\| \sum_\rho \psi_\rho^{(4)} \right\|_2^2 &\lesssim \#\left(\mathcal{F}\right) pR^{-\frac{1}{2}} \log R \\ \left\| \sum_\Theta \Psi_\Theta^{(4)} \right\|_2^2 &\lesssim \#\left(\mathcal{F}\right) \frac{R^{-\frac{1}{2}}}{p} \log R \end{aligned} \tag{3}$$

We also observe that, since the rectangles $\rho^* \in \mathcal{F}(\rho)$ can intersect at most along their boundary (if they are adjacent), $\psi_\rho^{(4)} \leq C$.

Define $H_\rho = \sum_{\rho^* \in \mathcal{F}(\rho)} G_\rho^{\rho^*}$. Recall that $G_\rho^{\rho^*} = b_{\rho^*} \widehat{\chi_\beta f(-x)}$. It follows that

$$\widehat{H_\rho} = \widehat{\sum_{\rho^* \in \mathcal{F}(\rho)} G_\rho^{\rho^*}} = \sum_{\rho^* \in \mathcal{F}(\rho)} \widehat{b_{\rho^*}} * \chi_\beta f(x) = \chi_\beta f(x) * \sum_{\rho^* \in \mathcal{F}(\rho)} \widehat{b_{\rho^*}}.$$

Note here that $\beta = \beta(\rho)$ and *supp* $\widehat{b_{\rho^*}} = \rho$, for all $\rho^* \in \mathcal{F}(\rho)$. Therefore *supp* $\widehat{H_\rho} = \rho + \beta \subset 2\rho$. We also set $H_\Theta = \sum_{\rho \subset \Theta} H_\rho$ and

$$P = \left(\sum_\Theta |H_\Theta|^2 \right)^{\frac{1}{2}}.$$

When we write $\rho \subseteq \Theta$ we mean that the corresponding $\beta(\rho)$ has radial projection onto the unit circle contained in the arc Θ.

Lemma 7.

$$\|P\|_4 \lesssim h(\log R)^{\frac{1}{4}} \left(\#\left(\mathcal{F}\right) pR^{-\frac{1}{2}} \right)^{\frac{1}{4}}.$$

Proof. For each ρ, $\widehat{H_\rho}$ is supported in 2ρ. By Lemma 3 we have

$$\|P\|_4 \lesssim \left\| \left(\sum_\rho |H_\rho|^2 \right)^{\frac{1}{2}} \right\|_4.$$

On the other hand,

$$|H_\rho| \leq \sum_{\rho^* \in \mathcal{F}(\rho)} h\phi_{\rho^*}^{(4)}.$$

Therefore

$$\begin{aligned}\|P\|_4 &\lesssim h\left\|\sum_{\rho}\left(\sum_{\rho^*\in\mathcal{F}(\rho)}\phi_{\rho^*}^{(4)}\right)^2\right\|_2^{\frac{1}{2}}\\ &= h\left\|\sum_{\rho}\left(\psi_{\rho}^{(4)}\right)^2\right\|_2^{\frac{1}{2}}\\ &\lesssim h\left\|\sum_{\rho}\psi_{\rho}^{(4)}\right\|_2^{\frac{1}{2}}\lesssim h(\log R)^{\frac{1}{4}}\left(\#(\mathcal{F})pR^{-\frac{1}{2}}\right)^{\frac{1}{4}}.\end{aligned}$$

□

We shall also need the following estimate for H_Θ. From the properties of $\mathcal{F}$ we know that $|G_\rho^{\rho^*}|\le h\phi_{\rho^*}^{(M)}$, so

$$|H_\Theta|\le\sum_{\rho\subset\Theta}|H_\rho|\le\sum_{\rho\subset\Theta}\sum_{\rho^*\in\mathcal{F}(\rho)}h\phi_{\rho^*}^{(4)}.$$

Let $\rho\subseteq\Theta$. From the proof of part 2 in Lemma 1, we know that all τ containing ρ^* have comparable angle. Choose C_1 so that every τ containing ρ^* belongs to $\mathcal{G}(\Theta)$.

We have

$$\sum_{\rho\subset\Theta}\sum_{\rho^*\in\mathcal{F}(\rho)}h\phi_{\rho^*}^{(M)}\lesssim h\sum_{\tau\in\mathcal{G}(\Theta)}\sum_{\rho^*\subseteq\tau}\phi_{\rho^*}^{(M)}.$$

Each τ has length C and width $\le C$. Each ρ^* has length 1 and width $\frac{1}{\sqrt{R}}$. Also, for each ρ the corresponding ρ^* are disjoint, so τ contains at most $\sqrt{R}$ rectangles ρ^* corresponding to the same ρ. The number of such ρ is at most $\sqrt{R}$, so each τ contains at most CR rectangles ρ^*. For every $\rho^*\in\mathcal{F}$ we have $\phi_{\rho^*}^{(M)}\le\phi_\tau^{(M)}$. Therefore

$$|H_\Theta|\le h\sum_{\tau\in\mathcal{G}(\Theta)}\sum_{\rho^*\subseteq\tau}\phi_{\rho^*}^{(M)}\lesssim hCR\sum_{\tau\in\mathcal{G}(\Theta)}\phi_\tau^{(M)}=hCR\Psi_\Theta^{(M)}.\tag{4}$$

From this point on we enter the main part of the proof. Every square of side length t introduced below is understood to be a translate of $[0,t)\times[0,t)$ by an element of $t\mathbb{Z}^2$, so these squares tile the plane. Observe that

$$\phi_\tau^{(4)}(x)\lesssim\left(\frac{\max(\theta,|x-y|)}{\theta}\right)^4\phi_\tau^{(4)}(y)$$

for every x and y, and from this it follows that

$$\Psi_\Theta^{(4)}(x)\lesssim\left(\frac{\max(\theta,|x-y|)}{\theta}\right)^4\Psi_\Theta^{(4)}(y).$$

Define an A-square to be a square Q of side length θ such that $\max_Q \sum_\Theta \Psi_\Theta^{(4)} \in [A, 2A]$. From the last inequality, if Q is an A-square then $\min_Q \sum_\Theta \Psi_\Theta^{(4)} \gtrsim \max_Q \sum_\Theta \Psi_\Theta^{(4)} \geq A$. From (3) we obtain

$$\begin{aligned}\text{number of } A\text{-squares } &= \frac{1}{A^2\theta^2} \sum_{Q\text{: the } Q \text{ is an } A\text{-square}} \int_Q A^2 \\ &\lesssim \frac{1}{A^2\theta^2} \sum_{Q\text{: the } Q \text{ is an } A\text{-square}} \int_Q \left|\sum_\Theta \Psi_\Theta^{(4)}\right|^2 \\ &\leq \frac{1}{A^2\theta^2} \left\|\sum_\Theta \Psi_\Theta^{(4)}\right\|_2^2 \leq \frac{1}{A^2\theta^2} \frac{\#\mathcal{F}}{p\sqrt{R}} \log R.\end{aligned}$$

Moreover, if x, y lie in a square Q of side length C, then

$$\left(\frac{\max\{\theta, |x-y|\}}{\theta}\right)^4 \lesssim \left(\frac{C}{R^{-\frac{1}{2}}}\right)^4 = CR^2,$$

and so

$$\max_Q \sum_\Theta \Psi_\Theta^{(4)} \lesssim R^2 \min_Q \sum_\Theta \Psi_\Theta^{(4)}.$$

If we restrict attention to the A-squares that intersect the support of μ, then all of them lie in one large square of side length C. On that square the function $\sum_\Theta \Psi_\Theta^{(4)}$ is bounded, so there are at most $C \log R$ values $A = 2^j$ corresponding to such 2^j-squares. We shall show that

$$\left|\int_{E_A} \sum_{\rho^* \in \mathcal{F}} G_\rho^{\rho^*} \, d\mu\right| \lesssim \left(B\mu\left(\mathbb{R}^2\right)\right)^{\frac{1}{2}} R^{\frac{1}{2}-\frac{\alpha}{2}+\epsilon}.$$

Here E_A is the union of all A-squares intersecting the support of μ.

Proposition 3. *There exists a radial function q such that $|q(x)| \leq C_N(1+|x|)^{-N}$ for every N, with the following property. If $t = (\theta R)^{-1}$, $q^t(x) = t^{-2}q(t^{-1}x)$, and $\overline{\mu}$ is the absolutely continuous measure $q^t * \mu$, namely*

$$\overline{\mu}(A) = \int_A \int_{\mathbb{R}^2} q^t(x-y) d\mu(y) dx$$

with density

$$\frac{d\overline{\mu}}{dx} = \int_{\mathbb{R}^2} q^t(x-y) d\mu(y),$$

then for every Θ and every square of side length greater than t,

$$\int_Q |H_\Theta| d\mu \lesssim \sum_{j\geq 0} 2^{-Mj} \int_{2^j Q} |H_\Theta| d\overline{\mu}. \tag{5}$$

Proof. We shall prove the existence of q so that (5) holds for squares Q of side length t. Let $\Lambda \in \mathbb{Z}$ be such that $\phi^{(4)}(x) \le C \sum_{j\ge 0} 2^{-j\Lambda} \chi_{2^j Q}$ and let k be a radial Schwartz function with $\widehat{k} = 1$ on a sufficiently large disk centered at the origin. Since t is fixed, the support of $\widehat{H_\Theta}$ lies inside a disk dilated by Ct^{-1} for some C. It follows that $\widehat{|H_\Theta|} \le C\widehat{|k^t| \cdot |H_\Theta|}$ and consequently $|H_\Theta| \lesssim |k^t| * |H_\Theta|$. Also

$$\begin{aligned}\int_Q |H_\Theta|\, d\mu \lesssim \int_Q |k^t| * |H_\Theta| &= \int_Q \int_{\mathbb{R}^2} |k^t|(x-y)|H_\Theta|(x) \frac{\phi_Q^{(4)}(y)}{\phi_Q^{(4)}(y)} d\mu(x)\, dy \\ &= \int_{\mathbb{R}^2} \int_Q \frac{|k^t|(x-y)}{\phi_Q^{(4)}(y)} d\mu(x) |H_\Theta|(x) \phi_Q^{(4)}(y)\, dy.\end{aligned}$$

Moreover, $\frac{1}{\phi_Q^{(4)}(y)} \le C\left(1 + t^{-1}\|y-x\|\right)^{\Lambda}$, so by setting $q(x) = (1+|x|)^{\Lambda}|k(x)|$, the function q decays as required, and furthermore

$$\int_Q |H_\Theta| d\mu \lesssim \int \phi_Q(y) |H_\Theta|(y) d\overline{\mu}(y) \lesssim \sum_{j\ge 0} 2^{-\Lambda j} \int_{2^j Q} |H_\Theta| d\overline{\mu}.$$

By simple geometric arguments, the inequality is first proved for the case where the squares have side length $2^j t$ with $j > 1$, and then the general case follows. ☐

It is also true that $\overline{\mu}\left(D_r\right) \lesssim CBr^{\alpha}$, where $B = \sup_x \sup_{r \ge R^{-1}} \frac{\mu\left(D(x;r)\right)}{r^{\alpha}}$. Indeed,

$$\begin{aligned}\overline{\mu}\left(D_r\right) &= \int_{\mathbb{R}^2} \int_{D(x_0;r)} q^t(x-y)\, d\mu(y)\, dx = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} q^t(x) \mathcal{X}_{D(x_0-y;r)}(x)\, d\mu(y)\, dx \\ &= \int_{\mathbb{R}^2} q^t(x) \int_{\mathbb{R}^2} \mathcal{X}_{D(x_0-x;r)}(y)\, d\mu(y)\, dx \le CBr^{\alpha}.\end{aligned}$$

It is also easy to see that $\overline{\mu}\left(\mathbb{R}^2\right) = C\mu\left(\mathbb{R}^2\right)$. We shall also need the following proposition.

Proposition 4.

$$\left\|\frac{d\overline{\mu}}{dx}\right\|_{\infty} \lesssim B(\theta R)^{2-\alpha}.$$

Proof.

$$\begin{aligned}\left|\frac{d\overline{\mu}}{dx}\right| &\leq \int \left|q^t(x-y)\right|\, d\mu(y) \leq (\theta R)^2 \int \left|q\big(\theta R(x-y)\big)\right|\, d\mu(y)\\
&\leq C_N(\theta R)^2 \int \frac{1}{\left(1+|\theta R(x-y)|\right)^N}\, d\mu(y)\\
&= C_N(\theta R)^2 \left[\int_{D(x,\frac{1}{\theta R})} \frac{1}{\left(1+|\theta R(x-y)|\right)^N}\, d\mu(y) + \sum_{k=1}^{\infty} \int_{\frac{2^{k-1}}{\theta R}\leq|x-y|\leq\frac{2^k}{\theta R}} \frac{1}{\left(1+|\theta R(x-y)|\right)^N}\, d\mu\right]\\
&\leq C_N(\theta R)^2 \left[\int_{D(x,\frac{1}{\theta R})} d\mu(y) + \sum_{k=1}^{\infty} \int_{\frac{2^{k-1}}{\theta R}\leq|x-y|\leq\frac{2^k}{\theta R}} \frac{1}{2^{(k-1)N}}\, d\mu\right]\\
&\leq (\theta R)^2 \left[B\frac{1}{(\theta R)^\alpha} + \sum_{k=1}^{+\infty} \frac{1}{2^{(k-1)N}}\mu\left(D\left(x,\tfrac{2^k}{\theta R}\right)\right)\right]\\
&\leq C_N(\theta R)^2 \left[B\frac{1}{(\theta R)^\alpha} + \sum_{k=1}^{+\infty} \frac{1}{2^{(k-1)N}} B\frac{2^{k\alpha}}{(\theta R)^\alpha}\right]\\
&\leq C_N(\theta R)^2 B(\theta R)^{-\alpha}\left[1+2^N\sum_{k=1}^{+\infty} 2^{k(\alpha-N)}\right] \leq C_N B\,(\theta R)^{2-\alpha}.\end{aligned}$$

□

Let Q be an A-square. Then

$$\left|\int_Q \sum_{\rho,\rho^*:\rho^*\in\mathcal{F}} G_\rho^{\rho^*}\, d\mu\right| \leq \sum_{\Theta}\int_Q |H_\Theta| d\mu = \sum_{\Theta:\max_Q \Psi_\Theta^{(4)}\geq R^{-\epsilon}} \int_Q |H_\Theta| d\mu + \sum_{\Theta:\max_Q \Psi_\Theta^{(4)}< R^{-\epsilon}} \int_Q |H_\Theta| d\mu.$$

For the second term, from (4) it follows that $|H_\Theta| \lesssim Rh\Psi_\Theta^{(M)} \lesssim Rh\left(\Psi_\Theta^{(4)}\right)^{\frac{M}{4}}$. If M is chosen sufficiently large, then

$$\sum_{\Theta:\max_Q \Psi_\Theta^{(4)}< R^{-\epsilon}} \int_Q |H_\Theta| d\mu \lesssim \mu(\mathbb{R}^2) h R^{1-\frac{M}{4}\epsilon} \leq \mu(\mathbb{R}^2) R^{-100}.$$

The number of Θ in the first term is at most $R^\epsilon A$, since $\Psi_\Theta^{(4)} \gtrsim R^{-\epsilon}$ and their sum is $\lesssim A$. Since $\theta \geq (R\theta)^{-1}$ and by Proposition 3,

$$\begin{aligned}\left|\int_Q \sum_{\rho,\rho^*:\rho^*\in\mathcal{F}} G_\rho^{\rho^*}\, d\mu\right| &\lesssim \sum_{j\geq 0} 2^{-\Lambda j}\int_{2^jQ} \sum_{\Theta:\max_Q \Psi_\Theta^{(4)}\geq R^{-\epsilon}} |H_\Theta| d\overline{\mu} + \mu(\mathbb{R}^2)R^{-100}\\
&\lesssim (AR^\epsilon)^{\frac{1}{2}}\sum_{j\geq 0} 2^{-\Lambda j}\int_{2^jQ} P d\overline{\mu} + \mu(\mathbb{R}^2)R^{-100}.\end{aligned}$$

We shall estimate the integral

$$\left|\int_{E_A} \sum_{\rho,\rho^*:\rho^*\in\mathcal{F}} G_\rho^{\rho^*}\, d\mu\right|$$

by summing over all A-squares. Let E_A^j be the 2^j-dilation of the union E_A of all A-squares. Since there are at most R such A-squares intersecting the support of μ, the second term contributes no more than $\mu(\mathbb{R}^2)R^{-99}$. From Proposition 4 we obtain

$$\left| \int_{E_A} \sum_{\rho,\rho^*:\rho^*\in\mathcal{F}} G_\rho^{\rho^*}\, d\mu \right| \lesssim (AR^\epsilon)^{\frac{1}{2}}\, 2\|P\|_4 \sum_j 2^{2j} 2^{-\Lambda j} \left\| \frac{d\overline{\mu}}{dx} \right\|_{L^{\frac{4}{3}}(E_A^j)} + \mu(\mathbb{R}^2)R^{-99}.$$

$$\lesssim (AR^\epsilon)^{\frac{1}{2}} (\log R)^{\frac{1}{4}}\, h \left(\#(\mathcal{F}) p R^{-\frac{1}{2}} \right)^{\frac{1}{4}} \cdot \sum_j 2^{(2-\Lambda)j} \mu\left(\mathbb{R}^2\right)^{\frac{1}{2}} (\theta R)^{\frac{2-\alpha}{4}}\ \overline{\mu}(E_A^j)^{\frac{3}{4}} + \mu(\mathbb{R}^2)R^{-99}.$$

Since for every x, $\overline{\mu}\left(D(x;r)\right) \lesssim CBr^\alpha$, every square of side length θ has $\overline{\mu}$-measure at most $B\theta^\alpha$. We also showed that the number of A-squares is at most $C\frac{\#(\mathcal{F})}{p\sqrt{R}} \log R\theta^{-2}A^{-2}$, so

$$\overline{\mu}\left(E_A\right) \lesssim \log R \frac{\#(\mathcal{F})}{p\sqrt{R}} \theta^{-2} A^{-2} B \theta^\alpha.$$

Equivalently,

$$p \lesssim \#(\mathcal{F}) B \theta^{-2} R^{-\frac{1}{2}} A^{-2} \overline{\mu}\left(E_A\right)^{-1} \theta^\alpha \log R.$$

A 2^j-dilate of an A-square has $\overline{\mu}$-measure at most $CB2^{j\alpha}\theta^\alpha$, and therefore

$$p \lesssim \#\left(\mathcal{F}\right) B \theta^{-2} R^{-\frac{1}{2}} A^{-2} \overline{\mu}\left(E_A^j\right)^{-1} 2^{j\alpha} \theta^\alpha \log R.$$

Using also that $h \lesssim R^{\frac{1}{4}} \left(\#\left(\mathcal{F}\right)\right)^{-\frac{1}{2}}$, we obtain

$$\left| \int_{E_A} \sum_{\rho,\rho^*:\rho^*\in\mathcal{F}} G_\rho^{\rho^*}\, d\mu \right| \lesssim$$

$$(\log R)^{\frac{1}{4}} (AR^\epsilon)^{\frac{1}{2}} \cdot \frac{R^{\frac{1}{4}}}{\left(\#(\mathcal{F})\right)^{\frac{1}{2}}} \cdot \sum_j \left(\left(\#\left(\mathcal{F}\right)\right)^2 B \theta^{-2} R^{-1} A^{-2} \overline{\mu}\left(E_A^j\right)^{-1} 2^{j\alpha} \theta^\alpha \log R \right)^{\frac{1}{4}} \cdot 2^{(2-\Lambda)j} B^{\frac{1}{4}} (\theta R)^{\frac{2-\alpha}{4}}\ \overline{\mu}(E_A^j)^{\frac{3}{4}}$$

$$+ \mu(\mathbb{R}^2)R^{-99}$$

$$\lesssim (\log R)^{\frac{1}{2}} R^{\frac{\epsilon}{2}} \sum_j 2^{\left(2-\Lambda+\frac{\alpha}{4}\right)j} B^{\frac{1}{2}} \overline{\mu}\left(E_A^j\right)^{\frac{1}{2}} R^{\left(\frac{1}{2}-\frac{\alpha}{4}\right)}$$

$$\lesssim (\log R)^{\frac{1}{2}} R^{\frac{\epsilon}{2}} \left(B\mu\left(\mathbb{R}^2\right)\right)^{\frac{1}{2}} R^{\left(\frac{1}{2}-\frac{\alpha}{4}\right)}.$$

Finally, from (2) we have

$$\left| \int G\, d\mu \right| \le CR^\epsilon \left| \int_{E_A} \sum_{\rho,\rho^*:\rho^*\in\mathcal{F}} G_\rho^{\rho^*}\, d\mu \right| \le C (\log R)^{\frac{1}{2}} R^{\frac{3\epsilon}{2}} \left(B\mu\left(\mathbb{R}^2\right)\right)^{\frac{1}{2}} R^{\left(\frac{1}{2}-\frac{\alpha}{4}\right)}.$$

If f is a function supported on A_R and $\|f\|_{L^2} = 1$, then

$$\left| \int_{A_R} \widehat{\mu}(x) f(-x)\, dx \right| = \left| \int G\, d\mu \right| \le (\log R)^{\frac{1}{2}} \left(B\mu\left(\mathbb{R}^2\right)\right)^{\frac{1}{2}} R^{\frac{-3\epsilon+1+\alpha}{2}}.$$

Therefore

$$\frac{1}{R}\int_{A_R} |\widehat{\mu}(x)|^2 \, dx \le C_\epsilon B \mu\left(\mathbb{R}^2\right) R^{-\frac{\alpha}{2}+4\epsilon}.$$

This, by Lemma 6, gives

$$\int_{-\pi}^{\pi} \left|\widehat{\mu}\left(Re^{i\theta}\right)\right|^2 \, d\theta \le C_\epsilon B \mu\left(\mathbb{R}^2\right) R^{-\frac{\alpha}{2}+4\epsilon}. \tag{6}$$

If we assume that μ is a measure supported on the unit disk and has α-dimensional energy 1, then, as in Lemma 5, we can find measures μ_j with $1 \le j \lesssim \log R$ such that $\mu = \sum_j \mu_j$ and

$$\mu_j\left(\mathbb{R}^2\right) \sup_x \sup_{r \ge R^{-1}} \frac{\mu_j\left(D(x,r)\right)}{r^\alpha} \lesssim 1.$$

For each one of these measures, (6) holds:

$$\int_{-\pi}^{\pi} \left|\widehat{\mu}_j\left(Re^{i\theta}\right)\right|^2 \, d\theta \le C_\epsilon \sup_x \sup_{r \ge R^{-1}} \frac{\mu_j\left(D(x,r)\right)}{r^\alpha} \mu_j\left(\mathbb{R}^2\right) R^{-\frac{\alpha}{2}+4\epsilon} \lesssim C_\epsilon R^{-\frac{\alpha}{2}+4\epsilon}.$$

Finally,

$$\begin{aligned} \int_{-\pi}^{\pi} \left|\widehat{\mu}\left(Re^{i\theta}\right)\right|^2 d\theta &= \int_{-\pi}^{\pi} \left|\sum_j \widehat{\mu_j}\left(Re^{i\theta}\right)\right|^2 d\theta \\ &\lesssim \log R \sum_j \int_{-\pi}^{\pi} \left|\widehat{\mu_j}\left(Re^{i\theta}\right)\right|^2 d\theta \\ &\lesssim C_\epsilon \left(\log R\right)^2 R^{-\frac{\alpha}{2}+4\epsilon} \lesssim C_\epsilon R^{-\frac{\alpha}{2}+6\epsilon}. \end{aligned}$$

This yields the theorem.

References

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