Trabajo fin de Máster

Carleson measures for Hardy spaces and Bergman spaces

Christos Chatzifountas

Director del trabajo: Profesor D. Daniel Girela Álvarez
Universidad de Málaga

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Introduction

Carleson measures were introduced in the early 1960s by L. Carleson [2, 3] to characterize the interpolating sequences in the algebra of bounded analytic functions in the open unit disc and to give a solution to the corona problem. Carleson measures play a very important role in many questions in complex analysis, harmonic analysis, operator theory, partial differential equations, etc.

In this work we shall be concerned with analytic functions in the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. The algebra of all bounded analytic functions in $\mathbb{D}$ will be denoted by H^∞. For $0 < p < \infty$, the Hardy space H^p consists of those functions f which are analytic in $\mathbb{D}$ and for which the integral means

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}$$

are bounded for $r \in (0, 1)$. We have $H^\infty \subset H^q \subset H^p$, whenever $0 < p < q < \infty$.

The Bergman space A^p $(0 < p < \infty)$ is the set of those functions f which are analytic in $\mathbb{D}$ and belong to $L^p(\mathbb{D})$.

If $f \in H^\infty$ and $\{a_k\}_{k=1}^\infty$ is a sequence of points in $\mathbb{D}$ then the sequence $\{f(a_k)\}_{k=1}^\infty$ belongs to the spaces ℓ^∞ of bounded sequences. Carleson considered the problem of describing the **universal interpolation sequences for $\mathbf{H}^\infty$**. These are the sequences $\{a_k\}_{k=1}^\infty \subset \mathbb{D}$ with the property that for every bounded sequence $\{w_k\}_{k=1}^\infty$ there exists an H^∞ function f such that $f(a_k) = w_k$ for all k. This can be stated as follows:

Given a sequence $\{a_k\}_{k=1}^\infty \subset \mathbb{D}$, let $T_\infty : H^\infty \to \ell^\infty$ be the linear operator defined by $T_\infty(f) = \{f(a_k)\}_{k=1}^\infty$. Then $\{a_k\}_{k=1}^\infty$ is a universal interpolation sequence for H^∞ if and only if $T_\infty(H^\infty) = \ell^\infty$. Carleson obtained a characterization of these sequences. He proved that they are the **uniformly separated sequences**. A sequence $\{a_k\}_{k=1}^\infty \subset \mathbb{D}$ is said to be a uniformly separated sequence if there exists $\delta > 0$ such that

$$\prod_{\substack{j=1 \\ j\neq k}} \left| \frac{a_k - a_j}{1 - \bar{a}_j a_k} \right| \geq \delta, \qquad \text{k=1,2,...}$$

The universal interpolation problem has a natural generalization to H^p $(0 < p < \infty)$ if we consider a kind of weighted interpolation:

If $0 < p < \infty$ and $f \in H^p$ then

$$(1-|z|)^{1/p}|f(z)| \to 0, \quad \text{as } |z| \to 1.$$

Thus, fixing $p \in (0, \infty)$ and a sequence $\{a_k\}_{k=1}^{\infty}$ of points in $\mathbb{D}$ with $|a_k| \to 1$, as $k \to \infty$, if we consider the linear operator T_p which assigns to each $f \in H^p$ the sequence $\{(1-|a_k|^2)^{1/p} f(a_k)\}_{k=1}^{\infty}$ we have that the range of this operator is contained in the space c_0 of sequences which tend to zero. On the other hand, T_p need not map H^p into ℓ^p, even if $\{a_k\}$ satisfies the Blaschke condition. We consider the problem of characterizing the sequences $\{a_k\}$ for which $T_p(H^p) = \ell^p$. It turns out that they are the same for all p, they are precisely the uniformly separated sequences.

A part of the proof of the interpolation problem just mentioned consisted in showing that if $\{a_k\}$ is a uniformly separated sequence and $f \in H^p$ $(0 < p < \infty)$ then

$$\left(\sum_{n=1}^{\infty} (1-|a_k|)|f(a_k)|^p \right)^{1/p} \leq C\|f\|_{H^p}.$$

In other words, if we let μ be the discrete measure in $\mathbb{D}$ defined by

$$\mu = \sum_{k=1}^{\infty} (1-|a_k|)\delta_{a_k}$$

where δ_{a_k} is the Dirac-δ with mass at a_k, then H^p is (continuously) contained in $L^p(d\mu)$.

It is natural to ask what other measures μ in $\mathbb{D}$ have this property. One trivial example is the Lebesgue area measure in $\mathbb{D}$ but there are many others and they can be characterized very neatly:

A positive Borel measure μ in $\mathbb{D}$ will be said to be a **Carleson measure** if there exists a constant $A \geq 0$ such that

$$\mu(S) \leq Ah$$

for every set S of the form

$$S = \{re^{it} : 1-h \leq r < 1, \theta \leq t \leq \theta + h\},$$

with $h \in (0,1]$ and $\theta \in \mathbb{R}$. Carleson proved that, for every $p \in (0, \infty)$,

$$H^p \subset L^p(d\mu) \;\Leftrightarrow\; \mu \text{ is a Carleson measure.}$$

This concept can be generalized: Suppose that X is a space of analytic functions in the disc and $p \in (0, \infty)$, and let μ be a positive Borel measure in $\mathbb{D}$. We say that μ is a p-Carleson measure for X if $X \subset L^p(d\mu)$. Carleson result can be stated saying that the p-Carleson measures for H^p $(0 < p < \infty)$ are precisely the Carleson measures.

Duren [4] extended Carleson result showing that, for $0 < p < q$, the q-Carleson measures for H^p are the so called "q/p-Carleson measures", that is, those positive Borel measure μ in $\mathbb{D}$ for which there exists a constant $A \geq 0$ such that

$$\mu(S) \leq A h^{q/p}$$

for every set S of the form

$$S = \{re^{it} : 1 - h \leq r < 1, \theta \leq t \leq \theta + h\},$$

with $h \in (0, 1]$ and $\theta \in \mathbb{R}$ as above. Since then a great deal of work has been devoted to the problem of characterizing the Carleson measures for distinct spaces X of analytic functions in $\mathbb{D}$ as Carleson measures arise in many questions involving analytic function spaces. In particular, they play a very important role studying the boundedness and compactness of operators acting between them.

This work is devoted to present the characterizations of the q-Carleson measures for the Hardy spaces H^p and the Bergman spaces A^p, $0 < p, q < \infty$. This will be done in Chapters 4 and 5. In order to do so we start presenting in Chapter 1 the basic results in the theory of these spaces which will be needed in our work.

Luecking [16, 18] developed a method to obtain a characterization of the q-Carleson measures for the space A^p in the case $p \leq q$ which brings into play the pseudohyperbolic metric. This is a metric which is more natural than the euclidean one for problems in the disc and then we devote Chapter 2 to study some basic results about this metric which will be needed later. Chapter 3 is devoted to the prove Carleson interpolation theoren. Finally, Chapter 6 is devoted to present some examples and simple applications, showing that a number of important classical results in the theory of Hardy spaces and Bergman spaces can be thought as Carleson-measure-type results and presenting the characterization of the space $BMOA$ in terms of Carleson measures, a result which is essential in Fefferman's duality theorem: $(H^1)^\star = BMOA$.

We finish this introduction noticing that, as usual, throughout this work we shall be using the convention that $C_{p,\alpha,q,\beta}\dots$ will denote a positive constant which depends only upon the displayed parameters $p, \alpha, q, \beta \dots$ (which sometimes will be omitted) but does not necessarily remain the same at different occurrences.

Chapter 1

Basic theory of Hardy spaces and Bergman spaces

In this chapter we shall present some of the basic facts of the theory of Hardy spaces and Bergman spaces which will be used in our work. We mention [5] and [9] as a general references for the theory of Hardy spaces and [6] and [14] for that of Bergman spaces.

1.1 Hardy spaces and their properties

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$ and let $\mathcal{H}ol(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ (endowed with the topology of uniform convergence on compact subsets). If $0 \leq r < 1$ and $f \in \mathcal{H}ol(\mathbb{D})$, we set

$$\begin{aligned} M_p(r,f) &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p}, \quad \text{if } 0 < p < \infty, \\ M_\infty(r,f) &= \max_{|z|=r} |f(z)|. \end{aligned}$$

These means $M_p(r,f)$ increase with r. For $0 < p \leq \infty$, the Hardy space H^p consists of those functions f, analytic in $\mathbb{D}$, for which

$$\|f\|_{H^p} = \sup_{0 \leq r < 1} M_p(r,f) = \lim_{r \to 1} M_p(r,f) < \infty.$$

Then we see that H^∞ is the class of bounded analytic functions in the disc. It is clear that $H^p \supset H^q$, whenever $0 < p < q \leq \infty$. For $1 \leq p \leq \infty$, $\| \cdot \|_{H^p}$ is a norm in H^p and $(H^p, \| \cdot \|_{H^p})$ is a Bancah space.

Each function $f \in H^p$ has a radial limit

$$f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$$

for almost every θ, and the boundary function has the properties $f(e^{i\theta}) \in L^p$ and $\log|f(e^{i\theta})| \in L^1$. Moreover $\|f\|_{H^p} = \|f(e^{i\theta})\|_{L^p(\mathbb{T})}$. Identifying each function $f \in H^p$ with the boundary-values function $f(e^{i\theta})$, we can view H^p as a subspace of $L^p(\mathbb{T})$.

The zeros z_k of f, repeated according to multiplicity, satisfy the *Blaschke condition*

$$\sum_{k=1}^{\infty}(1-|z_k|) < \infty$$

which ensures the convergence of the *Blaschke product*

$$B(z) = z^m \prod_{k=1}^{\infty} \frac{|z_k|}{z_k} \frac{z_k - z}{1-\bar{z}_k z}$$

This analytic function B has precisely the same zeros as f counted according to multiplicity. The non-vanishing function f/B again belongs to H^p and has the same norm as f. Blaschke products have the property that $B \in H^\infty$. More precisely, we have $|B(z)| < 1$ for all z, and $|B(e^{i\theta})| = 1$ a.e.

Any function $\phi \in H^\infty$ with $|\phi(e^{i\theta})| = 1$ a.e. is called an *inner function*. Blaschke products are inner functions. Another type of inner function is

$$S(z) = \exp\left\{-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\right\} \quad ,$$

where μ is a finite positive Borel measure in $\partial\mathbb{D}$ singular with respect to the Lebesgue measure. These functions S are called *singular inner functions*. The most general inner function is a product $e^{i\gamma}B(z)S(z)$, where $e^{i\gamma}$ is a unimodular constant, B is a Blaschke product, and S is a singular inner function.

Each function $f \in H^p$ has the canonical factorization $f = \phi F$, where ϕ is an inner function and F is an *outer function*, with the structure

$$F(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|f(e^{it})|\, dt\right\}$$

Functions in Hardy spaces have nice growth properties. For instance, if $0 < p < \infty$ and $f \in H^p$, then

$$|f(z)| = o((1-|z|)^{-1/p}), \quad \text{as } |z| \to 1. \tag{1.1}$$

Also, if $0 < p < q \leq \infty$ and $f \in H^p$ then

$$M_q(r, f) = o\left((1-r)^{1/q-1/p}\right), \qquad \text{as } r \to 1. \tag{1.2}$$

To see that the exponent is best possible, one need only consider functions of the form $f(z) = (1-z)^{-a}$ with $a > 0$, which belong to H^p if and only if $a < 1/p$. However, we remark that the last conclusion can be slightly sharpened to say that if $0 < p < q \leq \infty$ and $f \in H^p$ as above then

$$\int_0^1 (1-r)^{\lambda\alpha-1}\{M_q(r,f)\}^\lambda\, dr < \infty, \qquad \alpha = 1/p - 1/q,$$

for all $\lambda \geq p$. This result was proved by Hardy and Littlewood [11] and can be found in Chapter 5 of [5].

Information is available about the Taylor coefficients a_n of functions $f(z) = \sum_{n=1}^\infty a_n z^n$ in a Hardy space H^p. The case $p = 2$ is very simple:

$$f \in H^2 \text{ if and only if } \{a_n\} \in l^2,$$

that is, $f \in H^2$ if and only if $\sum_{n=1}^\infty |a_n|^2 < \infty$. The Hausdorff-Young inequalities give the following generalization:

If $1 < p \leq 2$, and q is the conjugate index $(1/p + 1/q = 1)$, then $f \in H^p$ implies $\{a_n\} \in l^q$; while $\{a_n\} \in l^p$ implies $f \in H^q$.

A theorem of Hardy and Littlewood provides another generalization: If $0 < p \leq 2$, and $f \in H^p$ then $\sum_{n=1}^\infty n^{p-2}|a_n|^p < \infty$. On the other hand, if $2 \leq p < \infty$ and $\sum_{n=1}^\infty n^{p-2}|a_n|^p < \infty$ then $f \in H^p$.

There is also some information in the case $p < 1$: Another theorem of Hardy and Littlewood asserts that if $f \in H^p$ for $0 < p \leq 1$, then $a_n = o(n^{1/p-1})$. All of these results are best possible.

Important result in the theory of Hardy spaces are also the Litlewood-Paley identities which relate the H^2-norm of a given function $f \in \mathcal{H}ol(\mathbb{D})$, with certain integrals which involve the integral means of order two of the derivative f' of f. This identities can be proved using either Parseval equality or Green's theorem (see [10, pp. 97–98]).

Theorem 1 (Littlewood-Paley identities). *If g is an analytic function in $\mathbb{D}$ then:*

$$2\int_{\mathbb{D}} |g'(z)|^2 \log\frac{1}{|z|}\, dA(z) = \|g - g(0)\|_{H^2}^2,$$

and

$$\begin{aligned}\int_{\mathbb{D}}(1-|z|^2)|g'(z)|^2\,dA(z) \leq& \|g-g(0)\|_{H^2}^2\\ \leq& C\int_{\mathbb{D}}(1-|z|^2)|g'(z)|^2\,dA(z)\end{aligned}$$

for a certain constant C independent of g.

Finally, we recall for $1 \leq p < \infty$ the dual space $(H^p)^\star$ is isometrically isomorphic to the quotient space $L^q(\mathbb{T})/H^q$, where q is the exponent conjugate of p. Furthermore, each $\phi \in (H^p)^\star$ can be expressed in the form

$$\phi(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\overline{g(e^{it})}\,dt$$

for a unique function $g \in H^q$ in the case $1 < p < \infty$, and for a function $g \in L^\infty$ in the case $p = 1$.

This representation of linear functionals can be applied to study extremal problems in Hardy spaces. In particular, it allows to discuss the following problem of "minimal interpolation": Let $z_1, z_2, \ldots, z_n$ be n distinct point in $\mathbb{D}$. The problem is to find an H^p-function ($1 \leq p \leq \infty$) of minimal norm which takes prescribed values

$$f(z_j) = w_j, \quad j = 1, 2, \ldots, n,$$

at the given points. Let

$$B(z) = \prod_{j=1}^{n}\frac{z-z_j}{1-\overline{z_j}z}, \quad B_k(z) = \prod_{j\neq k}\frac{z-z_j}{1-\overline{z_j}z}.$$

Then using the just mentioned "duality relation" it is not difficult to prove

$$\min_{f\in H^p, f(z_j)=w_j}\|f\|_{H^p} = \max_{f\in H^q, \|f\|_{H^q}\leq 1}\left|\sum_{j=1}^{n}\frac{w_j(1-|z_j|^2)}{B_j(z_j)}f(z_j)\right|. \tag{1.3}$$

We remark that it is possible to give a further step showing that $(H^p)^\star = H^q$, $1 < p < \infty$, and $(H^1)^\star = BMOA$ but this will not be needed in our work.

There is an analogue of Hardy spaces, for harmonic functions. If $u(z)$ is harmonic in the unit disc, we use the same definition of the integral means as the one given for an analytic function f, that is, we define $M_p(r, u)$, just with u and f switching places. That way we go one step further and define the *harmonic Hardy space* h^p.

As in the case of H^p, for $p \geq 1$ the means $M_p(r, u)$ increase with r and whenever $u \in h^p$ $(1 \leq p \leq \infty)$ the radial limit

$$u(e^{i\theta}) = \lim_{r \to 1} u(re^{i\theta})$$

exists for almost every θ. However, we remark that these facts do not remain true for $p < 1$.

As one might suspect the two Hardy spaces are connected. For a harmonic functions u let $\tilde{u}$ be the *harmonic conjugate*, the unique function that $\tilde{u}(0) = 0$ and $(u + i\tilde{u})(z)$ is analytic. If $p > 1$ and $u \in h^p$ a theorem of Marcel Riesz gives us that $\tilde{u}$ is also in h^p and the conjugate operator $u \to \tilde{u}$ is bounded,

$$\|\tilde{u}\|_{h^p} \leq C_p \|u\|_{h^p}$$

and hence

$$u \in h^p \;\Rightarrow\; u + i\tilde{u} \in H^p.$$

The opposite direction is just an application of the triangle inequality.

For $p = 1$ the situation changes:

$$\text{The implication } u \in h^1 \;\Rightarrow\; \tilde{u} \in h^1 \text{ does not hold.}$$

However, we have

$$u \in h^1 \;\Rightarrow\; \tilde{u} \in h^p, \quad 0 < p < 1.$$

It is also known that if $u \in L\, LogL$ then $\tilde{u} \in h^1$.

If $p < 1$ nothing can be said. The conjugate function $\tilde{u}$ of a function $u \in h^p$ $(0 < p < 1)$ may not belong to h^q for any positive q. An example is presented in Chapter 4 of [5].

1.2 Bergman spaces and their properties

For $0 < p \leq \infty$, the Bergman space A^p consists of all functions f analytic in the unit disc for which the area integral $\int_{\mathbb{D}} |f(z)|^p \, dA(z)$ is finite (here $dA(z)$ denotes the normalized area measure in $\mathbb{D}$ so that $A(\mathbb{D}) = 1$). In terms of integral means, the condition is that $\int_0^1 \{M_p(r, f)\}^p \, dr < \infty$. Immediately we see that $H^P \subset A^p$. In fact, it is true that

$$H^p \subset A^{2p}, \quad 0 < p < \infty.$$

This is a simple corollary of the integral means result (2), with $\lambda = q = 2p$.

If $M_p(r,f) = O\big((1-r)^{-a}\big)$ for some $a < 1/p$, then it is clear that $f \in A^p$. On the other hand,

$$f \in A^p \quad \Rightarrow \quad M_p(r,f) = o\left((1-r)^{-1/p}\right) \quad \text{ as } r \to 1. \tag{1.4}$$

For a proof, observe that if $f \in A^p$ then, using the fact that $M_p(\rho, f)$ is an increasing function of ρ, we obtain

$$(1-r)\{M_p(r,f)\}^p \leq \int_r^1 \{M_p(\rho,f)\}^p \, d\rho \to 0, \quad \text{ as } r \to 1.$$

In analogy with the fact that the function $(1-z)^\lambda$ is in H^p for any $\lambda < 1/p$ it can be shown that $(1-z)^\lambda$ is in A^p whenever $\lambda < 2/p$.

Using (1.4) and a standard result on integral means (see [5, Theorem 5.9])we deduce that

$$f \in A^p \quad \Rightarrow \quad M_\infty(r,f) = O\big((1-r)^{-2/p}\big). \tag{1.5}$$

In fact, it has been shown that "O" can be replaced by "o", and this estimate is then best possible. More generally:

If $f \in A^p$ then $M_q(r,f) = o\big((1-r)^{1/q-2/p}\big)$, for all $q \geq p$,

and the estimate is best possible. (Compare with (1.2)). If $q \leq p$, nothing better than $M_q(r,f) = o\big((1-r)^{-1/p}\big)$ can be said for arbitrary functions $f \in A^p$.

The derivative of a function in a Hardy space need not belong to any Hardy space. An example is the following: Set $f(z) = \sum_{k=0}^{\infty} \frac{1}{k^2} z^{2^k}$ $(z \in \mathbb{D})$. Then $f \in H^\infty$ but f' is given by a lacunary series whose sequence of coefficients is not in ℓ^2. Then (see [24]) it follows that f' lies in no Hardy space. The situation is different for Bergman spaces. Indeed, using (1.4) and a standard argument based on Cauchy's integral formula gives that if $f \in A^p$ then $M_p(r,f') = O\left((1-r)^{-\frac{1}{p}-1}\right)$. Then it follows that

$$f \in A^p \quad \Rightarrow \quad f' \in A^q, \quad \text{ for all } q < \frac{p}{p+1}.$$

There are much more differences between Bergman and Hardy functions. Functions in Bergman spaces need not have radial limits: We have already seen that an estimate of the form $M_p(r,f) = O\big((1-r)^{-a}\big)$ places f in A^p. It is known (see [5, Theorem 5.10])) that no growth estimate of the form $M_\infty(r,f) = O(\psi(r))$ whith $\phi(r) \to \infty$ can ensure that f has a radial limit on any set of positive measure. A typical example is the series

$$f(z) = \sum_{k=0}^{\infty} z^{2^k} = 1 + z^2 + z^4 + z^8 + \dots$$

It can be shown that $M_\infty = O\left(\log \frac{1}{1-r}\right)$, so that $f \in A^p$ for every $p < \infty$. However, as a lacunary series whose sequence of coefficients is not in ℓ^2, f has a radial limit almost nowhere.

As in the case of Hardy spaces, it is possible to give fairly precise conditions on the Taylor coefficients of a function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ in order that it belongs to a Bergman space A^p. In analogy with the fact that $f \in H^2$ if and only if $\{a_n\} \in l^2$, one can say that

$$f \in A^2 \;\Leftrightarrow\; \sum_{n=1}^{\infty} (n+1)^{-1} |a_n|^2 < \infty.$$

The following results also hold:

- If $f \in A^p$ and $0 < p < 1$ then $a_n = o\big(n^{2/p-1}\big)$.
- If $f \in A^p$ and $1 < p < \infty$ then $a_n = o\big(n^{1/p}\big)$.

Neither estimate can be improved.

The Bergman-space analogue of the Hardy-Littlewood theorem on Hardy spaces is somewhat more informative. It says

(a) If $0 < p \le 2$, then

$$\sum_{n=1}^{\infty} n^{-1} |a_n|^p < \infty \quad \Rightarrow \quad f \in A^p \quad \Rightarrow \quad \sum_{n=1}^{\infty} n^{p-3} |a_n|^p < \infty.$$

(b)

$$\sum_{n=1}^{\infty} n^{p-3} |a_n|^p < \infty \quad \Rightarrow \quad f \in A^p \quad \Rightarrow \quad \sum_{n=1}^{\infty} n^{-1} |a_n|^p < \infty.$$

All implications are best possible.

Finally, there is an analogue of the Hausdorff-Young theorem, found by Horowitz. Let $1 < p < \infty$ and let $q = p/(p-1)$ be its conjugate exponent.

(a) If $1 < p \le 2$, then

$$f \in A^p \quad \Rightarrow \quad \sum_{n=1}^{\infty} n^{1-q} |a_n|^q < \infty.$$

(b) If $2 \leq p < \infty$, then

$$\sum_{n=1}^{\infty} n^{1-q} |a_n|^q < \infty \quad \Rightarrow \quad f \in A^p.$$

Using summation by parts, Horowitz [15] obtained sufficient conditions for membership in A^p in terms of the growth of the sums

$$S_N^{(q)} = \sum_{n=1}^{N} |a_n|^q.$$

(a) If $1 < p < 2$, and $a < 2/p$, then

$$S_N^{(2)} = O\big(N^a\big) \quad \Rightarrow \quad f \in A^p$$

(b) If $2 < p < \infty$, $q = p(p-1)$ is the conjugate exponent, and $a < q - 1$, then

$$\sum S_N^{(2)} = O\big(N^a\big) \quad \Rightarrow \quad f \in A^p.$$

Indirect information about the coefficients of A^p functions can be gleaned from the study of coefficient multipliers of A^p into A^q. These are the complex sequences $\{\lambda_n\}$ with the property that $\sum_{n=1}^{\infty} \lambda_n a_n z^n$ is in A^q whenever $\sum_{n=1}^{\infty} a_n z^n$ is in A^p. Recent work has developed an extensive theory of coefficient multipliers, parallel to the well-established theory for the Hardy spaces [5].

The question of characterizing the sequences of zeros of functions in Bergman spaces is much more complicated than the corresponding one for Hardy spaces. Certainly, any Blaschke sequence is an A^p-zero sequences but there are funcions in A^p whose sequences of zeros are not Blaschke products. The first examples were given by Horowitz [15] who also showed that (contrary with what happens for Hardy spaces) the A^p-zero sequences vary with p.

Chapter 2

The pseudo-hyperbolic metric

The theory of Bergman spaces makes abundant use of the pseudohyperbolic metric, a metric which is more natural than the euclidean, for problems in the disc. In particular, the problem of characterizing Carleson measures for the Bergman spaces (and also for some other related spaces) is easier if we make use of the pseudohyperbolic metric. Consequently, in this chapter we collect some of the basic results about the pseudohyperbolic metric which will be needed in our work.

2.1 Definition and basic properties

For $w \in \mathbb{D}$ we let φ_w denote the Möbius transformation defined by

$$\varphi_w(z) = \frac{w - z}{1 - \bar{w}z}.$$

Then φ_w is a conformal mapping from the unit disc onto itself. We remark that φ_w is involutive, $\varphi_w(\varphi_w(z)) = z$, and interchanges 0 and w.

Let us denote by $\mathrm{Aut}(\mathbb{D})$ the group of all conformal mappings from $\mathbb{D}$ onto itself. It turns out that all the elements of $\mathrm{Aut}(\mathbb{D})$ are Möbius transformations. Indeed, we have

$$\mathrm{Aut}(\mathbb{D}) = \{\lambda\varphi_w : w \in \mathbb{D}, |\lambda| = 1\}.$$

Through Möbius transformations we define the *pseudo-hyperbolic metric* ρ by

$$\rho(z, w) = |\varphi_w(z)|, \quad z, w \in \mathbb{D}.$$

Note that $\rho(z, 0) = |z|$. The Schwarz-Pick theorem (the conformally invariant version of Schwarz's lemma) can be stated as follows:

If f is an analytic function in the unit disc $\mathbb{D}$ with $f(\mathbb{D}) \subset \mathbb{D}$ then

$$\rho(f(z), f(w)) \leq \rho(z, w), \quad z, w \in \mathbb{D}.$$

Furthermore, if equality holds for some pair of points $z, w \in \mathbb{D}$ with $z \neq w$ then $f \in \text{Aut}(\mathbb{D})$ and in such a case equality holds for every $z, w \in \mathbb{D}$.

It is not hard to verify that ρ is a true metric on $\mathbb{D}$. It is clear that $0 \leq \rho(0, w) < 1$, with $\rho(z, w) = 0$ if and only if $z = w$. The symmetry property $\rho(w, z) = \rho(z, w)$ is immediate. The triangle inequality

$$\rho(z, \zeta) \leq \rho(z, w) + \rho(z, \zeta)$$

is less obvious, and will be proved in a stronger form.

By direct calculations we can see that the following relations hold

(i) φ_w is an isometry with respect to the pseudo-hyperbolic metric

$$\rho(\varphi_w(z), \varphi_w(\zeta)) = \rho(z, \zeta) \qquad \text{for all z, } \zeta \in \mathbb{D}. \tag{2.1}$$

(ii) The quantities $1 - |z|^2$ and $1 - |\varphi_w(z)|^2$ are related by the equation

$$1 - |\varphi(z)|^2 = \frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \bar{w}z|^2}.$$

Property (i) says equivalently that ρ is Möbius invariant. Property (ii) is a basic identity that will be invoked repeatedly.

We will show that the inequalities

$$\frac{|\rho(z, w) - \rho(w, z)|}{1 - \rho(z, w)\rho(w, \zeta)} \leq \rho(w, \zeta) \leq \frac{\rho(z, w) + \rho(w, \zeta)}{1 + \rho(z, w)\rho(w, \zeta)}$$

hold for any points z, ζ, w in $\mathbb{D}$ also verifying that way, the triangle inequality for ρ. Since ρ is Möbius invariant, it suffices to assume that $w = 0$. The inequality then reduces to

$$\frac{||z| - |\zeta||}{1 - |z||\zeta|} \leq \left| \frac{z - \zeta}{1 - \bar{z}\zeta} \right| \leq \frac{||z| + |\zeta||}{1 + |z||\zeta|}$$

Also by radial symmetry, we may suppose that ζ is real and positive, and that $|z| = r < \zeta$. Then φ_ζ maps the circle $|z| = r$ onto a circle symmetric with respect to the real axis, intersecting the positive real axis at the two points $\frac{\zeta - r}{1 - r\zeta}$ and $\frac{\zeta + r}{1 + r\zeta}$. In particular,

$$\frac{\zeta - r}{1 - r\zeta} \leq |\varphi_\zeta(z)| < \frac{\zeta + r}{1 + r\zeta}$$

which proves the inequality.

It follows from the strong form of the triangle inequality that the estimates $\rho(z,w) \leq r$ and $\rho(z,\zeta) \leq s$ imply

$$\rho(z,\zeta) \leq \frac{r+s}{1+rs}$$

This is is true because the function $f(x,y) = \frac{x+y}{1-xy}$ attains a maximum value in the rectangle $[0,r] \times [0,s]$ at the point (r,s).

Another calculation shows that

$$\varphi_w'(z) = -\frac{1-|w|^2}{(1-\bar{w}z)^2}$$

so that

$$|\varphi_w'(z)| = \frac{1-|\varphi_w(z)|^2}{1-|z|^2}, \tag{2.2}$$

by the property (ii) above. This last identity says that the hyperbolic density $\frac{|ds|}{1-|z|^2}$ is Möbius invariant.

The expression

$$|\varphi_w'(z)|^2 = \frac{(1-|w|^2)^2}{(1-\bar{w}z)^4}$$

is the Jacobian that arises when performing a change of variables of type $\zeta = \varphi_w(z)$ in an integral. Thus for $f \in L^1(\mathbb{D})$ and Ω a measurable subset of $\mathbb{D}$.

$$\begin{aligned}\int_\Omega f(w)\,dA(w) = \int_{\varphi_w(\Omega)} f(\varphi_w(z))|\varphi_w{}'(z)|^2\,dA(z)\\ =(1-|w|^2)^2 \int_{\varphi_w\Omega} f(\varphi_w(z))|1-\bar{w}z|^{-4}\,dA(z).\end{aligned}$$

Another variant of this formula is

$$Bf(z) = \int_{\mathbb{D}} f(z)\frac{(1-|\zeta|^2)^2}{|1-\bar{\zeta}|^4} = \int_{\mathbb{D}} f(\varphi_\zeta(w))\,d\sigma(w).$$

The operator B is known as the Berezin transform. It arises often in the theory of Bergmam spaces, for instance in describing their Carleson measure

2.2 The pseudo-hyperbolic discs

The pseudo-hyperbolic disc with center w and radius r, $0 < r < 1$ is

$$\Delta(w,r) = \{z : \rho(z,w) < r\}.$$

It is clear that $\Delta(w,r) \subset \mathbb{D}$. Moreover, by the Möbius invariance of ρ, we have

$$\Delta(w,r) = \varphi_w(\Delta(0,r)) = \varphi_w(\{z : |z| < r\},$$

and so $\Delta(w,r)$ is a true Euclidean disc, since a Möbius transformation preserves circles. However, its Euclidean centre and radius are different from a and r unless $w = 0$. In order to compute its (normalized) area $|\Delta(w,r)|$, we make a change variables as follows to obtain

$$\begin{aligned} |\Delta(w,r)| &= \int_{\Delta(w,r)} dA(z) = \int_{\Delta(w,r)} |\varphi_w'|^2 \, dA(z) = \int_{\Delta(0,r)} \frac{(1-|w|^2)^2}{|1-\bar{w}z^4} dA(z) \\ &= r^2(1-|w|^2)^2 \int_{\mathbb{D}} \frac{dA(\zeta)}{|1-\bar{w}r\zeta|^4} = \left(\frac{r(1-|w|^2)}{1-r^2|w|^2}\right)^2 \end{aligned}$$

since

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{dA(z)}{|1-\bar{w}r\zeta|^4} = \|k_\beta\|_2^2 = \frac{1}{(1-|\beta|^2)^2}$$

where $k_\beta(z) = (1-\bar{\beta}z)^{-2}$ is the Bergman kernel function of $\mathbb{D}$.

It is now apparent that the Euclidean radius of the pseudo-hyperbolic disc $\Delta(w,r)$ is

$$R = \frac{r(1-|w|^2)}{1-r^2|w|^2} \tag{2.3}$$

as it's area is equal to R^2. The Euclidean centre of $\Delta(w,r)$ is

$$\gamma = \frac{(1-r^2)w}{1-r^2|w|^2}. \tag{2.4}$$

To see this, take $w > 0$ and observe that the disc $\Delta(w,r)$ is symmetric with respect to the real axis, while it's boundary circle crosses at the two points determined by

$$\frac{w-x}{1-wx} = \pm r, \qquad \text{or } x = \frac{w+r}{1+wr} \text{ and } \frac{w-r}{1-wr}.$$

The center is therefore

$$\gamma = \frac{1}{2}\left(\frac{w+r}{1+wr} + \frac{w-r}{1-wr}\right) = \frac{w(1-r^2)}{1-w^2r^2}$$

and the more general formula follows by a rotation. The euclidean radius can be calculated directly as

$$R = \frac{1}{2}\left(\frac{|w|+r}{1+|w|r} - \frac{|w|-r}{1-|w|r}\right) = \frac{r(1-|w|^2)}{1-|w|^2r^2}.$$

Any pair of pseudo-hyperbolically concentric circles are at constant pseudo-hyperbolic distance form each other. More precisely, if two circles C_1 and C_2 are defined by

$\rho(w, z) = r_1$ and $\rho(w, z) = r_2$, then each point $z_1 \in C_1$ has pseudo-hyperbolic distance

$$\min_{z_2 \in C_2} \rho(z_1, z_2) = \rho(r_1, r_2)$$

from the circle C_2. To prove this, it suffices to take $w = 0$, because of the Möbius invariance of the pseudo-hyperbolic metric. Then the problem reduces to showing by elementary calculus that $\rho(r_1, r_2 e^{i\theta})$ is smallest for $\theta = 0$.

The hyperbolic area of a measurable subset $\Omega \subset \mathbb{D}$ is defined by

$$a(\Omega) = \int_\Omega \frac{dA(z)}{(1 - |z|^2)^2}.$$

It is easily seen to be Möbius invariant. Indeed, if $w = \varphi_w(z)$, then

$$a(\Omega) = \int_{\varphi_w(\Omega)} \frac{dA(\zeta)}{(1 - |\zeta|^2)^2} = \int_\Omega |\varphi_w'(z)|^2 \frac{dA(z)}{(1 - ||\varphi_w(z)|^2|)^2} = \int_\Omega \frac{dA(z)}{(1 - |z|^2)^2}$$

by (2). As a consequence, a pseudo-hyperbolic disc $\Delta(w, r)$ has hyperbolic area

$$a(\Delta(w, r)) = a(\Delta(0, r)) = \frac{1}{\varphi} \int_0^r \int_0 2\pi \frac{s}{(1 - s^2)^2} \, d\theta \, ds = \frac{r^2}{1 - r^2}$$

Thus the hyperbolic area of $\Delta(w, r)$ depends only on the radius r and not on the centre w.

2.3 Some lemmas

Here we record several lemmas will be needed in the following chapters. The first two are basic facts about the pseudo-hyperbolic metric. The third one is a Covering lemma. The fourth is a kind of uniform subharmonic property with respect to pseudohyperbolic discs.

Lemma 1. *Any two points z and w in the same pseudohyperbolic disc $\Delta(a, r)$ satisfy an inequality of the form*

$$\frac{1}{C} \le \frac{1 - |z|}{1 - |w|} \le 2 \frac{1 - |z|^2}{1 - |w|^2} < C \tag{2.5}$$

Where C is a constant depending only on r.

Proof. By symmetry, it is enough to prove the upper inequality. Observe first that if z and ζ belong to the same pseudohyperbolic disc $\Delta(w, r)$, then by the strong form of the triangle inequality,

$$\rho(z, \zeta) \le \frac{\rho(z, w) + \rho(\zeta, z)}{1 + \rho(z, w)\rho(w, \zeta)} < \frac{2r}{1 + r^2} = s < 1.$$

On the other hand, the above property (ii) of the Möbius function φ_w states that

$$1 - \rho(z,\zeta)^2 = \frac{(1-|\zeta|^2)(1-|z|^2)}{|1-\bar{\zeta}z|}$$

which implies that

$$\frac{1-|z|^2}{1-|\zeta|^2} < \frac{(1-|z|^2)^2}{(1-s^2)|1-\bar{\zeta}z|^2} = \left(\frac{1+r^2}{1-r^2}\right)^2.$$

However $|1-\bar{\zeta}z| > 1-|z| > \frac{1}{2}(1-|z|^2)$ for every pair of points z and w in $\mathbb{D}$, so the conclusion is that

$$\frac{1-|z|}{1-|\zeta|} < 2\frac{1-|z|^2}{1-|\zeta|^2} < 8\left(\frac{1+r^2}{1-r^2}\right)^2.$$

□

Lemma 2. *In each pseudohyperbolic disc $\Delta(a,r)$, the kernel function*

$$k_a(z) = (1-\bar{a}z)^{-2}$$

satisfies the sharp inequalities

$$\left(\frac{1-r|a|}{1-|a|^2}\right)^2 \le |k_a(z)| \le \left(\frac{1+r|a|}{1-|a|^2}\right)^2, \quad z \in \Delta(a,r).$$

The proof is a simple computation and we omit it.

Lemma 3. *For every r with $0 < r < 1$ there exist a positive integer N (depending only on r) and a sequence $\{z_n\} \subset \mathbb{D}$ such that*

(i) $\mathbb{D} = \bigcup_{k=1}^{\infty} \Delta(z_k, r)$.

(ii) Every $z \in \mathbb{D}$ belongs to at most N of the dilated discs $\Delta(z_k, R)$, where $R = \frac{1}{2}(1+r)$.

Proof. Let $\{B_k\}$ be a sequence of pseudohyperbolic discs of radius $r/3$ such that $\mathbb{D} = \bigcup_{k=1}^{\infty} B_k$. We shall select a subsequence $\{D_k\}$ of $\{B_k\}$ inductively. Take $D_1 = B_1$. Having chosen $D_1, D_2, \quad, D_n$ we let D_{n+1} be the first disc of the sequence $\{B_k\}$ which is disjoint from $D_1 \cup D_2 \cdots \cup D_n$. For each k, let a_k be the (pseudohyperbolic) center of D_k. Thus $D_k = \Delta(a_k, r/3)$ $(k = 1, 2, \dots)$.

We claim now that $\mathbb{D} = \bigcup_{k=1}^{\infty} \Delta(a_k, r)$. Indeed, if a point $a \in \mathbb{D}$ did not belong to anyone of the discs $\Delta(a_k, r)$, then by the triangle inequality the disc $\Delta(a, 2r/3)$ would be disjoint from all the discs $\Delta(a_k, r/3)$. But then we would have that a

belongs to some of the discs B_j, and another application of the triangle inequality implies that $B_j \subset \Delta(a, 2r/3)$. In particular, B_j must disjoint from all the discs D_k. This is impossible by the way the sequence $\{D_k\}$ has been constructed. Thus we have proved that $\mathbb{D} = \bigcup_{k=1}^{\infty} \Delta(a_k, r)$.

For any $z \in \mathbb{D}$, consider the set of points

$$E = \{\varphi_z(a_k) : z \in \Delta(a_k, R)\}.$$

By definition, each point $w \in E$ lies in the disc $\Delta(0, R)$. On the other hand, since the pseudohyperbolic metric is Möbius invariant, any two distinct elements $w_j = \varphi_z(a_j)$ and $w_k = \varphi_z(a_k)$ in E satisfy

$$\rho(w_j, w_k) = \rho(a_j, a_k) \geq \frac{1}{3} r,$$

since the discs D_j and D_k are disjoint. But

$$\rho(w_j, w_k) = \left| \frac{w_k - w_j}{1 - \bar{w}_j w_k} \right| \leq \frac{|w_k - w_j|}{1 - R^2},$$

so $|w_k - w_j| \geq \frac{1}{3} r(1 - R^2)$. In other words, the set E consists of pints in the disc $\Delta(0, R)$ that are at least a distance $\frac{1}{3} r(1 - R^2)$ apart in the Euclidean metric. This shows that E can contain no more than N points, where N is some integer depending only on r. $\square$

Lemma 4. *Suppose that* $0 < p \leq \infty$, $0 < r < 1$, *and* f *is an analytic function in* $\mathbb{D}$, *then:*

(i) For arbitrary $z \in \mathbb{D}$

$$|f(z)|^p \leq \frac{C_r}{|\Delta(z, r)|} \int_{\Delta(z,r)} |f(\zeta)|^p \, dA(z)$$

(ii) For any $a \in \mathbb{D}$ *and for all* $z \in \Delta(a, r)$

$$|f(z)|^p \leq \frac{C_r}{|\Delta(a, r)|} \int_{\Delta(a,R)} |f(\zeta)|^p \, dA(\zeta), \quad R = \frac{1}{2}(1 + r).$$

Proof of (i). By the subharmonicity of $|f|^p$, we have

$$|f(0)|^p \leq \frac{1}{|\Delta(0, r)|} \int_{\Delta(0,r)} |f(w)|^p \, dA(w).$$

Replace f by $f \circ \varphi_z$ and make a change of variable to obtain

$$\begin{aligned}|f(z)|^p &\leq \frac{1}{|\Delta(0,r)|}\int_{\Delta(0,r)} |(f\circ\varphi)(w)|^p\, dA(w)\\ &= \frac{1}{|\Delta(0,r)|}\int_{\Delta(z,r)} |f(w)|^p \frac{(1-|z|^2)^2}{|1-z\overline{w}|^4}\, dA(w).\end{aligned}$$

And the desired result follows using Lemma 1.

Proof of (ii). Bearing in mind the formula for the area of a pseudohyperbolic disc given before, it is easy to prove that

$$|\Delta(a,r)| \asymp |\Delta(a,R)| \asymp |\Delta(z, \frac{1}{2}(1-r))|, \quad z\in\Delta(a,r), \tag{2.6}$$

with constants depending only on r.

Also, by the triangle inequality, we have

$$z\in\Delta(a,r) \ \Rightarrow\ \Delta\left((z, \frac{1}{2}(1-r)\right) \subset \Delta(a,R). \tag{2.7}$$

Take $z\in\Delta(a,r)$. Using (i), (2.7), and (2.6), we obtain

$$|f(z)|^p \leq \frac{C_r}{\left|\Delta\left((z,\frac{1}{2}(1-r)\right)\right|}\int_{\Delta(a,R)} |f(w)|^p\, dA(w) \leq \frac{C_r}{|\Delta(a,r)|}\int_{\Delta(a,R)} |f(w)|^p\, dA(w)$$

as desired. $\square$

Chapter 3

The universal interpolation problem for Hardy spaces

3.1 Universal interpolation sequences

The more general interpolation problem for a space X of analytic functions in the unit disc would be to characterize the pair of sequences $\{z_k\} \in \mathbb{D}$ and $\{w_k\} \subset \mathbb{C}$ for which there exists a function $f \in X$ such that $f(z_k) = w_k$, for all k. For $X = H^\infty$ this was solved by Nevanlinna and Pick. Here we shall consider the problem of describing the universal interpolation sequences for H^∞, and some related problems for H^p $(p > 0)$.

A sequence $\{z_k\} \in \mathbb{D}$ will be said to be a universal interpolation sequence for H^∞ if for any sequence $\{w_k\} \in \ell^\infty$ there exists a function $f \in H^\infty$ such that $f(z_k) = w_k$, for all k. Notice that, by the Weierstrass uniqueness theorem, the problem does not make sense unless $|z_k| \to 1$, as $k \to \infty$.

In order to consider interpolation sequences for H^p, it is helpful to adopt a more abstract point of view:

Let $\mathcal{S}$ be the space of all sequences $\{w_k\}_{k=1}^\infty$ of complex numbers. Let $\{z_k\}_{k=1}^\infty$ be a sequence of points in $\mathbb{D}$ with $|z_k| \to 1$, as $k \to \infty$ and let $T_\infty : \mathcal{H}ol(\mathbb{D}) \to \mathcal{S}$ be the operator defined by $T_\infty(f) = \{f(z_k)\}_{k=1}^\infty$. Notice that $T_\infty(H^\infty) \subset \ell^\infty$ and that $\{z_k\}$ is a universal interpolation sequence for H^∞ if and only if $T_\infty(H^\infty) = \ell^\infty$. Carleson [2] characterized these sequences showing that they are precisely the so called *uniformly separated sequences* which will be defined later.

Now, we recall that if $0 < p < \infty$ and $f \in H^p$ then

$$(1 - |z|)^{1/p} |f(z)| \to 0, \quad \text{as } |z| \to 1.$$

Then, if $\{z_k\}_{k=1}^{\infty}$ is a sequence of points in $\mathbb{D}$ with $|z_k| \to 1$, as $k \to \infty$, as above and $0 < p < \infty$ we define $T_p : \mathcal{H}ol(\mathbb{D}) \to \mathcal{S}$ by

$$T_p(f) = \{(1 - |z_k|^2)^{1/p} f(z_k)\}_{k=1}^{\infty}.$$

We have just noticed that T_p maps H^p into the space c_0 of null sequences. We remark that $T_p(H^p)$ need not be contained in ℓ^p even if the sequence $\{z_k\}$ satisfies the Blaschke condition (we shall see an example in Chapter 6). However, it turns out that $T(H^p) = \ell^p$ if and only if $\{z_k\}$ is a universal interpolation sequence for H^∞.

3.2 Uniformly separated sequences. Carleson Theorem

Definition 1. *A sequence $\{z_k\}$ is said to be uniformly separated if there exists a number $\delta > 0$ such that*

$$\prod_{\substack{j=1 \\ j \neq k}} \left| \frac{z_k - z_j}{1 - \bar{z}_j z_k} \right| \geq \delta, \qquad k=1,2,\ldots$$

The operators T_p and the uniformly separated sequences are connected by the following theorem.

Theorem 2. *For $0 < p \leq \infty$, $T_p(H^p) = l^p$ if and only if $\{z_k\}$ is uniformly separated.*

We remark that this and the definitions give that the universal interpolation sequences for H^∞ are precisely the uniformly separated dequences.

We start considering the the Banach space case: $1 \leq p \leq \infty$.

It is trivial that T_∞ is bounded from H^∞ to ℓ^∞.

Suppose now that $p < \infty$ and that $T(H^p) \subset \ell^p$. Then T_p is also continuous from H^p into ℓ^p. This follows because T_p is a closed operator: Suppose $f_n \in H^p$, $f_n \to f$, and

$$T_p(f_n) \to \{(1 - |z_k|^2)^{1/p} f_n(z_k)\} \to w \in l^p$$

From Cantor's diagonal argument $T_p(f_n) \to T_p(f) = w$ which means that T_p is a closed operator for $0 < p < \infty$.

In the following we may treat $1/p$ as 0, whenever $p = \infty$. Suppose now that $T_p(H^p) = l^p$, and let

$$N^p = \{f \in H^p : f(z_k) = 0,\ k = 1, 2, 3, \ldots\}$$

denote the kernel of T_p. The quotient space H^p/N^p is a Banach space and T_p is one to one from H^p to H^p/N^p. Combining this with the fact that T_p is closed, the open mapping theorem dictates that T_p is also an isomorphism between these spaces. So for arbitrary $\{w\}$, there is $f \in H^p$ such that

$$(1-|z_k|^2)^{-1/p} f(z_k) = w_k, \qquad k = 1, 2, \dots$$

and because T_p^{-1} is bounded :

$$\|f\| \le M\|w\|$$

So, for $k = 1, 2, 3 \dots$, if we take the l^p sequence $0, 0, 0, \dots, 1, 0, 0 \dots$ with 1 on the k-th place, there exists a function $f_k \in H^p$ such that

$$f_k(z_j) = \begin{cases} \left(1-|z_k|^2\right)^{-1/p} & \text{if} \quad j = k, \\ 0 & \text{if} \quad j \ne k, \end{cases}$$

For $n > k$, let

$$F_{nk}(z) = f_k(z) \prod_{\substack{j=1 \\ j\ne k}} \frac{1-\bar{z}_j z_k}{z_k - z_j} \tag{3.1}$$

Clearly $F_{nk} \in H^p$ and $\|F_{nk}\|_{H^p} = \|f_k\|_{H^p} \le M$. On the other hand, using the relation (1.1) it follows that

$$|F_{nk}(z)| \le C\|F_{nk}\|_p \left(1-|z|^2\right)^{-1/p},$$

So

$$\left| \prod_{\substack{j=1 \\ j\ne k}} \frac{1-\bar{z}_j z_k}{z_k - z_j} \right| = |F_{nk}(z_k)|(1-|z_k|^2)^{1/p} \le \|F_{nk}\| \le CM.$$

And $\{z_k\}$ is uniformly separated.

To prove the other way around we give some notation first. Set

$$\begin{aligned} B_n(z) &= \prod_{j=1}^{n} \frac{z - z_j}{1 - z\bar{z}_j} \\ B_{nk}(z) &= \frac{1 - z\bar{z}_k}{z - z_k} B_n(z) = \prod_{\substack{j=1 \\ j\ne k}}^{n} \frac{z - z_j}{1 - z\bar{z}_j}, \qquad k \le n \\ b_{nk} &= B_{nk}(z_k). \end{aligned}$$

As in many other cases it is easier to prove the hypothesis for $p = 2$ first and then we will use it to generalize the result. So for $w = \{w_k\} \in l^2$ and a uniformly separated sequence $\{z_k\}$ we have to find $f \in H^2$ such that $T_2 f = w$. Let

$$g_{nk}(z) = (1-|z_k|^2)^{3/2}[B_n(z)]^2(z-z_k)^{-2},$$
$$f_n(z) = \sum_{k=1}^{n} w_k b_{nk}^{-2} g_{nk}(z).$$

Then

$$(1-|z_k|^2)^{1/2} f_n(z_k) = w_k \qquad k = 1, 2, \ldots, n,$$

and

$$\|f_n\|^2 = (f_n, f_n) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} w_k b_{nk}^{-2} \bar{w}_j \bar{b}_{nj}^{-2} (g_{nk}, g_{nj})$$

Now

$$\begin{aligned}(g_{nk}, g_{nj}) &= (1-|z_k|^2)^{3/2}(1-|z_j|^2)^{3/2} \int_0^{2\pi} \frac{1}{\left[(e^{i\theta}-z_k)(e^{-i\theta}-\bar{z}_j)\right]^2}\, d\theta \\ &= (1-|z_k|^2)^{3/2}(1-|z_j|^2)^{3/2} \int_{|\xi|=1} \frac{\xi^2 \bar{z}_j^{-2}}{\left[(\xi-z_k)(\xi-\bar{z}_j^{-1})\right]^2}\, d\xi.\end{aligned}$$

By virtue of the residue theorem the value of the last integral is

$$(1+z_j\bar{z}_k)(1-z_j\bar{z}_k))^{-3}.$$

Then it follows that

$$\begin{aligned}\left|(g_{nj}g_{nk})\right| &= \left|(1+z_j\bar{z}_k)(1-z_j\bar{z}_k)^{-3}(1-|z_k|^2)^{3/2}(1-|z_j|^2)^{3/2}\right| \\ &\le 2\big(1-|z_j|^2\big)\big(1-|z_k|^2\big) \\ &= 2\left(1-\left|\frac{z_k-z_j}{1-\bar{z}_j z_k}\right|^2\right)\end{aligned}$$

So

$$\begin{aligned}\sum_{j,k}\left|(g_{nj}g_{nk})\right| &= 2\log\left[\prod_{j,k}\left(\exp\left|2^{-1}(g_{(n,j)}g_{nk})\right|\right)\right] \\ &\le 2\log\left[\prod_{j,k}\left((1-\left|(g_{(n,j)}g_{nk})\right|2^{-1}\right)^{-1}\right] \\ &\le 2\log\left[\prod_{j,k}\left|\frac{1-\bar{z}_j z_k}{z_k-z_j}\right|^2\right] \\ &\le -4\log\delta\end{aligned}$$

Because for $0 < x < 2$, $e^{x/2} \leq (1 - x/2)^{-1}$. For the function f we have:

$$\begin{aligned}
\|f_n\|_2^2 &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} w_k b_{nk}^{-2}(g_{nk}, g_{nj})^{1/2}\bar{w}_j \bar{b_{nj}}^{-2}(g_{nk}, g_{nj})^{1/2} \\
&= \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{|w_k|^2}{|b_{nk}|}(g_{nk}, g_{nj}) + \sum_{k=1}^{\infty}\frac{|\bar{w}_j|^2}{|\bar{b_{nj}}|}(g_{nk}, g_{nj})\right) \\
&\leq \sum_{k=1}^{\infty}\frac{|w_k|^2}{|b_{nk}|}\sum_{j=1}^{\infty}(g_{nk}, g_{nj}) + \sum_{j=1}^{\infty}\frac{|\bar{w}_j|^2}{|\bar{b_{nj}}|}(-4\log\delta) \leq -8\delta^{-2}\log\delta\sum_{k=1}^{\infty}|w_k|^2 \\
&= C\|w\|^2
\end{aligned}$$

where we used the fact that $\{z_k\}$ being uniformly separated implies $b_k > \delta^2$. This implies that $\{f_n\}$ is a normal family and so there must exist an H^2 function f and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$, with $T_2(f) = w$ and $\|f\| \leq M\|w\|$.

It remains to show that $T_2(H^2) \subset l^2$. Let $w = w_k$ be an arbitrary l^2 sequence, and let $h_n(z)$ be the H_2 function of minimal norm such that

$$h_n(z_k) = (1 - |z_k|^2)^{-1/2} w_k$$

We have already proved the existence of at least one function with that property, which has norm less that $C\|w\|$. Whence

$$\|h\| \leq \|w\|$$

Appealing to the result on minimal interpolation (1.3) presented in Chapter 1, we therefore see that each $f \in H_2$ satisfies

$$\left|\sum_{k=1}^{n}\frac{w_k(1 - |z_k|^2)^{1/2}}{b_{nk}} f(z_k)\right| \leq \|h_n\|\|w\| \leq C\|w\|\|f\|$$

Taking the supremum over all $w \in l^2$ with $\|w\| = 1$, we conclude that

$$\|Tf\| = \left(\sum_{k=1}^{n}(1 - |z_k|^2)f(z_k)^2\right)^{1/2} \leq C\|f\|$$

Thus $T_f \in l^2$ for all $f \in H_2$. For the general case, if $f \in H^p$, then f admits a factorization $f = Bg$, where B is a Blaskche product and g is an non-vanishing H^p function. Thus

$$\begin{aligned}
\left(\sum_{k=1}^{n}(1 - |z_k|^2)f(z_k)^p\right)^{1/2} &\leq \sum_{k=1}^{n}(1 - |z_k|^2)(g(z_k))^p = \sum_{k=1}^{n}(1 - |z_k|^2)(g(z_k)^{p/2})^2 \\
&\leq C\|g^{p/2}\|_2^2 = C\|g\|_p^p = C\|f\|_p^p
\end{aligned}$$

And then we see that $T_p(H^p) \subset l^p$. To prove the other inclusion $l^p \subset T_p(H^p)$, $1 \leq p \leq \infty$, given $\{w_k\} \in l^p$, let $g_n(z)$ be the H^p function of minimal norm which satisfies

$$g_n(z_k) = (1-|z_k|^2)^{-1/p} w_k, \quad k = 1, 2, \ldots, n$$

Using again property (1.3) on Hardy spaces, we have

$$\|g_n\| = \left| \sum_{k=1}^{n} \frac{w_k(1-|z_k|^2)^{1/q}}{b_{nk}} f(z_k) \right|$$

for some $f \in H^q$ with $\|f\| = 1$ where q is the conjugate of p. Thus,

$$\|g_n\| \leq \delta^{-1} \left\{ \sum_{k=1}^{n} |w_k|^p \right\}^{1/p} \left\{ \sum_{k=1}^{n} (1-|z_k|^2)|f(z_k)|^q \right\}^{1/q} \leq \delta^{-1} C^{2/q} \|w\|$$

And the family of functions $\{g_n\}$ is bounded in H^p and therefore constitutes a normal family. Whence there is an H^p function g, and a subsequence $\{g_{n_j}\}$ such that $g_{n_j} \to g$ with $g(z_k) = w_k$. Thus $\{w_k\} \in T_p(H^p)$.

For $0 < p < 1$ we don't have a substitute for the relation (1.3) but, unexpectedly, there is a simply construction that gives a function $f \in H^p$ with $T_p(f) = w$.

Let

$$b_n(z) = \prod_{\substack{j=1 \\ j \neq k}} \frac{|z_k|}{z_k} \frac{z_j - z}{1 - z_j z}$$

and

$$g_k(z) = (1 - \bar{z}_k z)^{-2/p}.$$

Since $|b_k(z_k)| \geq \delta > 0$, the series

$$f(z) = \sum_{k=1}^{\infty} (1-|z_k|^2)^{1/p} w_k [b_k(z_k)]^{-1} g_k(z)$$

converges uniformly in each disc $|z| < R < 1$. Hence f is analytic in $|z| < 1$. Furthermore,

$$M_p^p(r, f) \leq \delta^{-p} \sum_{k=1}^{\infty} (1-|z_k|^2)|w_k|^p M_p^p(r, g_k) \geq \delta^{-p} \sum_{k=1}^{\infty} |w_k| \leq \infty,$$

providing that $f \in H^p$. It is immediate that $T_p(f) = w$. This shows that $T_p(H^p) = l^p$ if $\{z_k\}$ is uniformly separated

The converse can be proved by virtually the same argument that was used in the case $1 \leq p \leq \infty$. Of course in this case H^p and ℓ^p are no longer Banach spaces but they are *Fréchet spaces*. Hence the closed graph theorem and open mapping theorem are still available. We omit the details.

We close this chapter noticing that important examples of uniformly separated sequence are the exponential sequences: A sequence $\{a_k\} \in \mathbb{D}$ is said to be exponential if there exist a constant $\lambda < 1$ such that

$$1 - |a_{k+1}| \leq \lambda(1 - |a_k|), \quad k = 1, 2, \ldots.$$

The fact that an exponential sequence is uniformly separated can be proved by a simple computation (see [5, Chapter 9]).

The converse is not true in general but it is true for sequences lying in a radius: If $0 < a_k < 1$, for all k then $\{a_k\}$ is uniformly separated if and only if it is exponential.

Chapter 4

Carleson measures for Hardy spaces

A part of the proof of the interpolation problem treated in the preceding chapter consisted in showing that if $\{z_n\}$ is a uniformly separated sequence and $f \in H^p$ $(0 < p < \infty)$ then

$$\left(\sum_{n=1}^{\infty}(1-|z_n|)|f(z_n)|^p\right)^{1/p} \leq C\|f\|_{H^p}.$$

In other words, if we let μ be the discrete measure in $\mathbb{D}$ given by

$$\mu = \sum_{n=1}^{\infty}(1-|z_n|)\delta_{z_n}$$

where δ_{z_n} is the Dirac-δ with mass at z_n, then H^p is (continuously) contained in $L^p(d\mu)$.

In this chapter we shall deal with the question of characterizing the measures μ in $\mathbb{D}$ with this property.

Definition 2. *A positive Borel measure μ in $\mathbb{D}$ will be said to be a Carleson measure for H^p $(0 < p < \infty)$ if $H^p \subset L^p(d\mu)$.*

More generally we can introduce the following definition.

Definition 3. *Suppose that $0 < p, q < \infty$ and let μ be a positive Borel measure in $\mathbb{D}$. We say that μ is a q-Carleson measure for H^p if $H^p \subset L^q(d\mu)$.*

We remark that, using the closed graph theorem (and the fact that convergence in H^p implies uniform convergence on compact subsets), we see that:

If $H^p \subset L^q(d\mu)$ then the inclusion $i : H^p \to L^p(d\mu)$ is continuous.

Carleson [2] gave a characterization of the now called Carleson measures for H^p in terms of "Carleson boxes". Duren [4] extended this characterizing the q-Carleson measures for H^p for $p < q$ also in terms of "Carleson boxes". For $q < p$ a characterization is also known but it is of a different kind.

4.1 The case $p \leq q$

If $0 < h \leq 1$ and $\theta \in \mathbb{R}$ se set

$$S_{\theta,h} = \{re^{it} : 1 - h \leq r < 1, \theta \leq t \leq \theta + h\}.$$

A subset S of the disc will be said to be a Carleson box if it is of the form $S_{\theta,h}$ for some admissible pair (θ, h).

Definition 4. *Let μ be a positive Borel measure in $\mathbb{D}$. We say that μ is Carleson measure if there exists a positive constant A such that*

$$\mu(S_{\theta,h}) \leq Ah, \quad 0 < h \leq 1, \quad \theta \in \mathbb{R}.$$

The smallest constant A for which this is true is called de Carleson-constant of μ and will be denoted by $N(\mu)$.

More generally we can make the following definition.

Definition 5. *Suppose that $\alpha > 0$ and let μ be a positive Borel measure in $\mathbb{D}$. We say that μ is an α-Carleson measure if there exists a positive constant A such that*

$$\mu(S_{\theta,h}) \leq Ah^{\alpha}, \quad 0 < h \leq 1, \quad \theta \in \mathbb{R}.$$

With these definitions the 1-Carleson measures are simply the Carleson measures.

We remark that if μ is an α-Carleson measure for some $\alpha > 0$, then it is a finite measure. Also, since the constant functions belong to H^p, it is trivial that q-Carleson measures for H^p $(0 < p, q < \infty)$ are necessarily finite measures.

Theorem 3. *Suppose that $0 < p \leq q < \infty$ and let μ be a positive Borel measure in $\mathbb{D}$, then μ is a q-Carleson measure for H^p if and only if it is a q/p-Carleson measure.*

Proof of the necessity. Suppose that $0 < p \leq q < \infty$ and that μ is a q-Carleson measure for H^p. As noted above this implies that the inclusion mapping

$$i : H^p \to L^q(d\mu)$$

is continuous, that is, there exists a positive constant C such that

$$\left(\int_{\mathbb{D}} |f(z)|^q \, d\mu(z) \right)^{1/q} \leq C \|f\|_{H^p}, \quad \text{for all } f \in H^p. \tag{4.1}$$

Take $\theta \in \mathbb{R}$ and $h \in (0,1]$. Set $\rho = 1 - h$ and $z_0 = \rho e^{i\theta}$, and consider the H^p function g defined by

$$g(z) = (1 - \bar{z}_0 z)^{-2/p}, \quad z \in \mathbb{D},$$

whose H^p-norm is

$$\|g\|_{H^p} = (1 - \rho^2)^{-1/p}. \tag{4.2}$$

Now, a simple geometric argument shows that there exists an absolute constant A such that

$$|1 - \bar{z}_0 z| \leq Ah, \quad z \in S_{\theta,h}.$$

Using this, (4.1), and (4.2) we obtain

$$h^{-(2q)/p} \mu(S_{\theta,h}) \leq C \int_{S_{\theta,h}} |g(z)|^q \, d\mu(z) \leq C h^{-q/p},$$

and hence

$$\mu(S_{\theta,h}) \leq C h^{q/p}.$$

□

The sufficiency is more difficult. We shall prove it first in the case $p = 2 \leq q < \infty$ (the general case follows easily from this). We will show that (4.1) will follow from the fact that the hypothesis $\mu(S_{\theta,h}) \leq C h^{q/2}$ implies that a certain operator T is of "type $(2, q)$", and the theorem will be proved using Marzinkiewicz's interpolation theorem.

Hence, before embarking into the proof we shall introduce some concepts and results.

Definition 6. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces and let $\mathcal{M}(X)$ (respectively, $\mathcal{M}(Y)$) denote the class of all measurable (real or complex valued) and a.e. finite functions defined in X (respectively, defined in Y). Let S be a subspace of $\mathcal{M}(X)$ and let $T : S \to \mathcal{M}(Y)$ be an operator which is sublinear, that is, it satisfies*

$$T(f_1 + f_2)(x) \leq Tf_1(x) + Tf_2(x), \text{ for all } f_1, f_2 \in S \text{ and for almost every } x \in X.$$

Suppose that $1 \leq r \leq \infty$ and $1 \leq s \leq \infty$.

- *T is said to be of type (r,s) if it is defined in $L^r(d\mu)$ and there exists a positive constant M such that*

$$\|Tf\|_{L^s(d\nu)} \leq M\|f\|_{L^r(d\mu)}, \quad \text{for all } f \in L^r(d\mu).$$

- *If $s<\infty$, then T is said to be of weak type (r,s) if T defined in $L^r(d\mu)$ and there exists a positive constant M such that*

$$\mu\left(\{x \in X : |Tf(x)| > \lambda\}\right) \leq \left(\frac{M}{\lambda}\|f\|_{L^r(d\mu)}\right)^s, \quad \lambda > 0,$$

for every $f \in L^r(d\mu)$.

- *If $s=\infty$, T is said to be of weak type (r,s) if it is of type (r,s), that is, we define weak type (r,∞) as identical with type (r,∞).*

It is clear that type (r,s) implies weak type (r,s). For the sake of emphasis, operators of type (r,s) are also called of strong type (r,s). There exist lots of operators which are of weak type (r,s) without being of strong type (r,s). Some examples are the conjugation operator and the Hardy-Littlewood maximal operator which are of weak type $(1,1)$ but not of strong type $(1,1)$.

The following theorem of interpolation has lots of important applications. It can be found in [24, Chapter XII, Vol. II].

Theorem 4 (**Interpolation Theorem of Marzinkiewicz**). *Suppose that $0 \leq \beta_j \leq \alpha_j \leq 1$ $(j=1,2)$ and that $\beta_1 \neq \beta_2$. Suppose also that the sublinear operator T is simultaneously of weak types $(1/\alpha_1, 1/\beta_1)$ and $(1/\alpha_2, 1/\beta_2)$ with constants M_1 and M_2, respectively. Then T is of type (α,β) for any pair (α,β) with*

$$\alpha = (1-t)\alpha_1 + t\alpha_2, \quad \beta = (1-t)\beta_1 + t\beta_2 \quad (0<t<1)$$

and we have

$$\|Tf\|_{1/\beta} \leq K M_1^{1-t} M_2^t \|f\|_{1/\alpha}.$$

Specially important cases of Marzinkiewicz's theorem are the following:

- If $1 \leq p < q \leq \infty$ and T is of weak type (p,p) and of weak type (q,q) then it is of strong type (r,r) for every $r \in (p,q)$.

- If $1 \leq s \leq \infty$ and T is of weak type $(1,s)$ and of weak type (∞,∞) then it is of strong type $(2,2s)$.

Proof of the sufficiency in the case $p = 2 \leq q < \infty$.

Suppose that μ satisfies $\mu(S_{\theta,h}) \leq Ch^{q/2}$. For each $f \in H^2$ we have

$$|f(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} P_r(t-\theta)|f(e^{it})|dt, \quad z = re^{i\theta}.$$

Therefore, to show that μ is a q-Carleson measure for H^2 it will suffice to prove that

$$\left(\int_{\mathbb{D}} [u(z)]^q \, d\mu(z) \right)^{1/q} \leq C\|\varphi\|_2 \tag{4.3}$$

if u is the Poisson integral of a positive L^2-function φ.

With each point $z = re^{i\theta} \neq 0$ in $\mathbb{D}$ we associate the boundary arc

$$I_z = \{e^{it} : \theta - \frac{1}{2}(1-r) \leq t \leq \theta + \frac{1}{2}(1-r)\}.$$

Choose $\theta = 0$ if $z = 0$. Given $\phi \in L^1(\mathbb{T})$ we define

$$T\phi(z) = \sup_{\substack{I \supset I_z \\ |I| \leq 1}} \frac{1}{|I|} \int_I |\phi(t)| \, dt,$$

where the supremum is taken over all the arcs I containing I_z. It is clear that $T\phi$ is continuous in $\mathbb{D}$ whenever $\phi \in L^1(\mathbb{T})$, and then it follows that T is a sublinear operator from $L^1(\mathbb{T}, d\theta)$ into $\mathcal{M}(\mathbb{D}, d\mu)$.

Let $u(z)$ be the Poisson integral of $|\phi|$,

$$u(z) = \int_{-\pi}^{\pi} P(r, t-\theta)|\phi(t)| \, dt.$$

We will prove that

$$u(z) \leq C(T\phi(z) + \|\phi\|_1) \tag{4.4}$$

using a stopping time argument.

Fix $z \in \mathbb{D}$, $z = re^{i\theta}$, let $t_n = 2^{n-1}(1-r)$. Define

$$\omega_n = [-t_n, t_n], \quad n = 0, 1, 2, \ldots, N,$$

where N is the largest integer n such that $t_n < \frac{1}{2}$. Now define the sets

$$\begin{aligned} &G_0 = \omega_0, \qquad G_n = \omega_n \setminus \omega_{n-1} \qquad n = 0, 1, 2, \ldots, N \\ &G_{N+1} = [-\pi, \pi] \setminus \omega_N \end{aligned}$$

Then,

$$u(z) = \frac{1}{2\pi}\sum_{n=1}^{N+1}\int_{G_n} P(r,t)|\phi(t+\theta)|\,d\theta \leq \frac{1}{2\pi}\sum_{n=1}^{N+1} P(r,t_{n-1})\int_{G_n}|\phi(t+\theta)|\,d\theta$$

where $t_{-1}=0$. Now, since for $t \leq \pi$

$$\begin{aligned} P(r,t) = \frac{1-r^2}{1-2r\cos(t)+r^2} &= \frac{1-r^2}{(1-r)^2+2r(1-\cos(t))} = \frac{1-r^2}{(1-r)^2+4r\sin^2(\frac{t}{2})} \\ &\leq \frac{1-r^2}{(1-r)^2+4\pi^{-2}rt^2}, \end{aligned}$$

we find that

$$P(r,t_n) \leq \frac{C}{4^n(1-r)}, \quad \tfrac{1}{2} \leq r < 1.$$

Then it follows that for $\frac{1}{2} \leq r < 1$

$$\begin{aligned} u(z) \leq& \frac{C}{1-r}\sum_{n=0}^{N+1} 4^{-n}\int_{G_n}\phi(t+\theta)dt \\ \leq& C(\sum_{n=0}^{N+1} 2^{-n}T\phi(z) + \|\phi\|_1) \\ \leq& C_2(T\phi(z)+\|\phi\|_1) \end{aligned}$$

for a suitable constant C_2.

For $r < \frac{1}{2}$ we have trivially

$$u(z) \leq \frac{2}{\pi}\int_{-\pi}^{\pi}\phi(\theta+t) = 4\|\phi\|_1 \leq C_2(T\phi(z)+\|\phi\|_1)$$

Having chosen $C_2 > 4$. Thus (4.4) holds.

Now we shall prove that the operator $T : L^1(\mathbb{T}, d\theta) \to \mathcal{M}(\mathbb{D}, d\mu)$ is of weak type $(1, q/2)$. In order to do so we shall use the following covering lemma.

A covering lemma

Let A be a non-empty subset of $\mathbb{D}$ that does not contain an infinite sequence of points whose associated arcs I_z are pairwise disjoint. Then there exist a finite number of points $z_1, z_2, \ldots, z_m$ in A such that the corresponding I_{z_n} are pairwise disjoint and

$$A \subset \cup_{n=1}^{m}\{z : I_z \subset 5I_{z_n}\}$$

Proof. Let
$$\rho_1 = \inf\{|z| : z \in A\}$$
and choose $z_1 \in A$ such that $|z_1| < \frac{1}{2}(1+\rho_1)$. If $0 \in A$, choose $z_1 = 0$. Suppose that we have already chosen $z_1, z_2, \ldots z_{n-1}$. Inductively, define

$$\rho_n = \inf\{|z| : z \in A \,\text{and}\, I_z \cap I_j = \emptyset,\ j = 1, 2, 3, \ldots, n-1\}.$$

Then choose $z_n \in A$ so $I_{z_n} \cap I_{z_j} = \emptyset$ for $j = 1, \ldots, n-1$ and $|z_n| < \frac{1}{2}(1+\rho_n)$. By the hypothesis this process must terminate after the choice of a finite number m of points z_n. Now for each $z \in A$ we have $I_z \cap I_{z_n} \neq \emptyset$ for some n ($n = 1, \ldots, m$). Suppose n is the smallest such index. Then $|z| \geq \rho_n$ which gives

$$1 - |z| \leq 1 - \rho_n < 2(1 - |z|).$$

This together with the fact that I_z meets I_{z_n} implies that I_z is contained in $5I_{z_n}$, which proves the lemma. $\square$

Now we continue with the proof of the theorem: T is of weak type $(1, q/2)$.

For $s, \epsilon > 0$, define
$$E_s = \{z : T\phi(z) > s\}$$

$$A_s^\epsilon = \left\{z : \frac{1}{|I_z| + \epsilon}\int_{I_z} |\phi(t)|\, dt > s\right\}$$

$$B_s^\epsilon = \{z : I_z \subset I_w \text{ for some } w \in A_s^\epsilon\}$$

It is easy to check that the sets B_s^ϵ increase as $\epsilon \downarrow 0$ and that $E_s = \cup_{\epsilon>0} B_s^\epsilon$. Hence
$$\mu(E_s) = \lim_{\epsilon\to 0} \mu(B_s^\epsilon).$$

For any given $\epsilon > 0$ the set A_s^ϵ can have only a finite number of points z_n whose associated arcs I_{z_n} are pairwise disjoint. Indeed, otherwise we would have

$$\infty = s\sum_{n=1}^{\infty}(\epsilon + |I_n|) < \sum_{n=1}^{\infty}\int_{I_n} |\phi(t)| dt \leq 2\pi\|\phi\|_1, \tag{4.5}$$

a contradiction.

If the set E_s is empty then $\mu(E_s) = 0$. Otherwise, A_s^ϵ is nonempty for sufficiently small ϵ and then using the covering lemma we can choose points $z_1, z_2, \ldots, z_n \in A_s^\epsilon$ such that
$$A_s^\epsilon \subset \bigcup_{n=1}^{\infty}\{z : I_z \subset 5I_{z_n}\}$$

and the arcs I_{z_n} are pairwise disjoint. Then it follows easily that

$$B_s^\epsilon \subset \bigcup_{n=1}^{\infty}\{z : I_z \subset 5I_{z_n}\}. \tag{4.6}$$

Since the measure μ satisfies $\mu(S_{\theta,h}) \le Ch^{q/2}$, we obtain that

$$\mu(\{z : I_z \subset 5I_{z_n}\} \le C|I_{z_n}|^{\frac{q}{2}}, \quad \text{for all } n.$$

Using this, (4.6), and the definition of the set A_s^ϵ, we deduce that

$$\mu(B_s^\epsilon) < C\sum_{n=1}^{m}|I_{z_n}|^{\frac{q}{2}} \le C\left(\frac{\|\phi\|_1}{s}\right)^{q/2},$$

for all $\epsilon > 0$. This implies

$$\mu(E_s) = \lim_{\epsilon\to 0}\mu(B_s^\epsilon) \le Cs^{-\frac{q}{2}}\|\phi\|_1^{\frac{q}{2}}.$$

Thus, the operator T is weak of $(1, q/2)$. Since it is trivially of strong type (∞,∞), using the Marcinkiewicz's interpolation theorem we deduce that it is of strong type $(2, q)$, that is, there exists a constant C such that

$$\left(\int_{\mathbb{D}}|T\phi(z)|^q\,d\mu(z)\right)^{1/q} \le C\|\phi\|_2, \quad \text{for all } \phi \in L^2(\mathbb{T}),$$

which, using (4.4) and bearing in mind that $\|\phi\|_1 \le \|\phi\|_2$ and that μ is a finite measure, implies

$$\left(\int_{\mathbb{D}}u(z)^q\,d\mu(z)\right)^{1/q} \le C\left(\|T\phi\|_{L^q(d\mu)} + \|\phi\|_2\right) \le C\|\phi\|_2,$$

whenever $\phi \in L^2(\mathbb{T})$ and u is the Poisson integral of $|\phi|$. This shows that (4.3) holds whenever u is the Poisson integral of a positive L^2-function φ finishing the proof. □

Proof of the sufficiency in the case general case $0 < p \le q < \infty$.
Suppose that $0 < p \le q < \infty$ and μ satisfies

$$\mu\left(S_{\theta,h}\right) \le Ch^{q/p}, \quad \theta \in \mathbb{R},\ 0 < h \le 1.$$

By the case just proved, we see that

$$H^2 \textit{ is continuously contained in } L^{(2q)/p}(d\mu). \tag{4.7}$$

Take $f \in H^p$, then we can factor f in the form $f = Bg$ where B is a Blaschke product and g is an H^p-function without zeros and with $\|f\|_{H^p} = \|g\|_{H^p}$. Since g is never zero, we can define an analytic branch of $g^{\frac{p}{2}}$ which is an H^2-function with $\|g^{\frac{p}{2}}\|_{H^2} = \|f\|_{H^p}^{p/2}$. Also, since $|B| \le 1$, we have $|f(z)| \le |g(z)|$, for all z. Using these facts we obtain

$$\int_{\mathbb{D}} |f(z)|^q \, d\mu \le \int_{\mathbb{D}} |g(z)|^q \, d\mu = \int_{\mathbb{D}} |g^{p/2}(z)|^{(2q)/p} \, d\mu \le C \|g^{p/2}\|_{H^2}^{(2q)/p} = C\|g\|_{H^p}^q.$$

This shows that μ is a q-Carleson measure for H^p finishing the proof. □

4.2 The case $q < p$

Characterizations of the q-Carleson measures for H^p in the case $q < p$ have been obtained in several works (see, e. g., [19] and [20] and some other references mentioned there). We shall follow the arguments of [19]. The characterization we have in this case depends on integrability properties of a certain function $\tilde{\mu}$ associated to μ. This function $\tilde{\mu}$ is usually called the balayage of μ.

For $0 < \alpha < \frac{\pi}{2}$ and $\theta \in \mathbb{R}$ we let $\Gamma_\alpha(\theta)$ denote the interior of the convex hull of the disc $\{|z| < \sin\alpha\}$ and the point $e^{i\theta}$. Then $\Gamma_\alpha(\theta)$ is an "ice-cream domain" with vertex $e^{i\theta}$, of angle 2α, and symmetric with respect to the radius from the origin through $e^{i\theta}$. From now $\alpha \in (0, \pi/2)$ will be fixed.

Let χ_θ denote the characteristic function of the set $\Gamma_\alpha(\theta)$. We have:

Lemma 5. *There exists a positive constant A_α (depending only on α) such that*

$$\int_0^{2\pi} \chi_\theta(z) \, d\theta \le A_\alpha(1 - |z|), \quad z \in \mathbb{D}.$$

Proof. For simplicity, denote

$$I(z) = \int_0^{2\pi} \chi_\theta(z) \, d\theta$$

and we distinguish the two cases

$$(i)\, |z| < \sin\alpha, \quad (ii)\, |z| \ge \sin\alpha.$$

In case (i), $\chi_\theta(z) = 1$ for all θ and then $I(z) = 2\pi$.

In case (ii), a simple geometric argument shows that $\chi_\theta(z) = 0$ except for the θ's in an interval whose length is bounded by $A_\alpha(1 - |z|)$ for a certain positive constant A_α. Hence $I(z) \le A_\alpha(1 - |z|)$, an inequality which also holds in case (i). □

We shall use also the non-tangential maximal characterization of Hardy spaces: If F is a real or complex valued function defined in $\mathbb{D}$ and $\theta \in \mathbb{R}$, we set

$$N_\alpha(F)(\theta) = \sup\{|F(z)| : z \in \Gamma_\alpha(\theta)\}.$$

We have:

Theorem 5 (Non-tangential maximal characterization of Hardy spaces). *Suppose that $0 < p < \infty$ and $f \in \mathcal{H}ol(\mathbb{D})$. Then*

$$f \in H^p \text{ if and only if } N_\alpha(f) \in L^p(\mathbb{T}).$$

Furthermore, $\|f\|_{H^p} \asymp \|N_\alpha(f)\|_{L^p}$.

Now we can give a characterization of q-Carleson measures for H^p $(q < p)$.

Theorem 6. *Suppose that $0 < q < p < \infty$ and let μ be a positive Borel measure in $\mathbb{D}$. Then, μ is a q-Carleson measure for the space H^p if and only if the function $\tilde{\mu}$ defined by*

$$\tilde{\mu}(t) = \int_{\Gamma_\alpha(t)} \frac{d\mu(z)}{1-|z|}, \quad t \in \mathbb{R},$$

belongs to $L^{p/(p-q)}(\mathbb{T})$.

Proof. We start proving the sufficiency. Hence, suppose that the function $t \mapsto \int_{\Gamma_\alpha(t)} \frac{d\mu(z)}{1-|z|}$ belongs to $L^{p/(p-q)}(\mathbb{T})$ and take $f \in H^p$. Using Lemma 5 and Fubini's theorem we obtain

$$\begin{aligned}
\int_{\mathbb{D}} |f(z)|^q \, d\mu(z) &\asymp \int_{\mathbb{D}} \frac{|f(z)|^q}{1-|z|} \int_0^{2\pi} \chi_\theta(z) \, d\theta \, d\mu(z) \\
&= \int_0^{2\pi} \int_{\mathbb{D}} \frac{|f(z)|^q}{1-|z|} \chi_\theta(z) \, d\mu(z) \, d\theta \\
&= \int_0^{2\pi} \int_{\Gamma_\alpha(\theta)} \frac{|f(z)|^q}{1-|z|} \, d\mu(z) \, d\theta \\
&\le \int_0^{2\pi} N_\alpha(f)(\theta)^q \int_{\Gamma_\alpha(\theta)} \frac{d\mu(z)}{1-|z|} \, d\theta.
\end{aligned}$$

Now, using the non-tangential maximal characterization of H^q we see that $N_\alpha(f)^q \in L^{p/q}(\mathbb{T})$ which, using the fact that p/q and $p/(p-q)$ are conjugate exponents and our hypothesis, implies that

$$\int_0^{2\pi} N_\alpha(f)(\theta)^q \int_{\Gamma_\alpha(\theta)} \frac{d\mu(z)}{1-|z|} \, d\theta < \infty.$$

And, $f \in L^q(d\mu)$ as desired.

Now we turn to prove the necessity. So, let us assume that μ is a q-Carleson measure for H^p. Let ϕ be a positive function in $L^{\frac{p}{q}}(\mathbb{T})$ and let u be the Poisson integral of ϕ and $f = u + i\tilde{u}$. Since $p/q > 1$, we have that $u \in h^{p/q}$ and $f \in H^{p/q}$. Also, $\Re f = u > 0$ in $\mathbb{D}$. In particular, f never vanishes in $\mathbb{D}$ and then there is an analytic branch g of $f^{1/q}$ which is an H^p-function. We have

$$\int_{\mathbb{D}} u(z)\, d\mu(z) \le \int_{\mathbb{D}} |f(z)|\, d\mu(z) = \int_{\mathbb{D}} |g(z)|^q\, d\mu(z) \le C\|g\|_{H^p}^q = C\|f\|_{H^{p/q}} \le C\|\phi\|_{L^{p/q}}.$$

Now, writing u as the Poisson integral of ϕ and using the simple fact that

$$|1 - ze^{-it}| \asymp 1 - |z|, \quad z \in \Gamma_\alpha(t), \text{ (with constants depending on } \alpha \text{ only)},$$

we obtain

$$\begin{aligned}
C\|\phi\|_{L^{p/q}} &\ge \int_{\mathbb{D}} \int_0^{2\pi} \frac{1-|z|^2}{|1-ze^{-it}|^2} \phi(t)\, dt\, d\mu(z) \\
&= \int_0^{2\pi} \phi(t) \int_{\mathbb{D}} \frac{1-|z|^2}{|1-ze^{-it}|^2}\, d\mu(z)\, dt \\
&\ge \int_0^{2\pi} \phi(t) \int_{\Gamma_\alpha(t)} \frac{1-|z|^2}{|1-ze^{-it}|^2}\, d\mu(z)\, dt \\
&\ge C' \int_0^{2\pi} \phi(t) \int_{\Gamma_\alpha(t)} \frac{d\mu(z)}{1-|z|}\, dt \\
&= C' \int_0^{2\pi} \phi(t)\tilde{\mu}(t)\, dt.
\end{aligned}$$

Thus we have proved that $\int_0^{2\pi} \phi(t)\tilde{\mu}(t)\, dt < \infty$ for all positive ϕ in $L^{p/q}(\mathbb{T})$. By duality, this implies that $\tilde{\mu} \in L^{p/(p-q)}(\mathbb{T})$. $\square$

Chapter 5

Carleson measures for Bergman spaces

5.1 The case $p \leq q$

For $0 < p \leq q < \infty$ the q-Carleson measures for the Bergman space A^p were first characterized by Hastings in [12] who showed that they have a description similar to that of the corresponding ones for Hardy space.

Theorem 7. *Suppose that $0 < p \leq q < \infty$ and let μ be positive Borel measure in $\mathbb{D}$ and suppose tha $1 \leq p \leq q < \infty$. Then the following are equivalent*

(i) μ is a q-Carleson measure for the Bergman space A^p.

(ii) μ is a $(2q)/p$-Carleson measure.

Proof. $(i) \to (ii)$ Suppose that $0 < p \leq q < \infty$ and that μ is a q-Carleson measure for the space A^p. Then there exists a positive constant C such that

$$\left(\int_{\mathbb{D}} |f(z)|^q \, d\mu(z)\right)^{1/q} \leq C\|f\|_{A^p}, \quad \text{for all } f \in A^p. \tag{5.1}$$

Take $\theta \in \mathbb{R}$ and $h \in (0, 1]$. Set $\rho = 1 - h$ and $z_0 = \rho e^{i\theta}$, and consider the A^p-function $g = g_{\theta,h}$ defined by

$$g(z) = (1 - \bar{z}_0 z)^{-4/p}, \quad z \in \mathbb{D},$$

whose A^p-norm is

$$\|g\|_{A^p} \asymp (1 - \rho^2)^{-2/p}. \tag{5.2}$$

Now, we recall that there exists an absolute constant A such that

$$|1 - \bar{z}_0 z| \leq Ah, \quad z \in S_{\theta,h}.$$

Using this, (5.1), and (5.2) we obtain

$$h^{-(4q)/p}\mu(S_{\theta,h}) \leq C\int_{S_{\theta,h}} |g(z)|^q\, d\mu(z) \leq Ch^{-2q/p},$$

and hence

$$\mu(S_{\theta,h}) \leq Ch^{2q/p}.$$

$\square$

(ii) $\rightarrow$ *(i)*. Suppose that μ is a $(2q)/p$-Carleson measure and let $f \in A^p$. For $m = 0, 1, 2, \ldots$ and $k \in \{1, 2, \ldots, 2^{m+4}\}$ we set

$$T_{m,k} = \left\{re^{i\theta} : 1 - 2^{-m} \leq r < 1 - 2^{-m-1},\ \frac{2k\pi}{2^{m+4}} \leq \theta < \frac{2(k+1)\pi}{2^{m+4}}\right\}$$

and

$$z^{(m,k)} = (1 - 2^{-m})\exp\left(\frac{2(k+1/2)\pi i}{2^{m+4}}\right).$$

By direct calculations one can see that for every m, k,

$$\operatorname{diam}(T_{m,k}) \leq \frac{11}{16}2^{-m}.$$

Fix $T_{m,k}$. For every $z \in T_{(m,k)}$ take $\frac{6}{8}2^{-m} \leq \rho \leq \frac{7}{8}2^{-m}$ and apply Harnack's inequality on the disc of center $z^{(m,k)}$ and radius ρ to the subharmonic function $|f|^p$ to get that

$$\begin{aligned}
|f(z)|^p &= \left|f\left(z^{(m,k)} + |z - z^{(m,k)}|e^{i\theta_0}\right)\right|^p \\
&\leq \frac{1}{2\pi}\frac{\rho + |z - z^{(m,k)}|}{\rho - |z - z^{(m,k)}|}\int_0^{2\pi} |f(z^{(m,k)} + re^{i\theta})|^p\, d\theta \\
&\leq C\int_0^{2\pi} |f(z_{(m,k)} + re^{i\theta})|^p\, d\theta.
\end{aligned}$$

Hence, for $z \in T_{m,k}$,

$$\begin{aligned}
|f(z)|^p &= C4^{-m}\int_{\frac{6}{8}2^{-m}}^{\frac{7}{8}2^{-m}} \rho|f(z)|^p\, d\rho \\
&\leq C4^{-m}\int_{\frac{6}{8}2^{-m}}^{\frac{7}{8}2^{-m}}\int_0^{2\pi} \rho|f(z_{(m,k)} + re^{i\theta})|^p\, d\theta\, d\rho \\
&\leq C4^{-m}\int_{U_{(m,k)}} |f(w)|^p dA(w),
\end{aligned}$$

where $U_{(m,k)} = w : |w - z_{(m,k)}| \leq \frac{7}{8}2^{-m}$. So

$$\begin{aligned}
\int_{\mathbb{D}} |f(z)|^q \, d\mu &= \sum_{m,k} \int_{T_{m,k}} |f(z)|^{p\frac{q}{p}} \, d\mu \\
&\leq \sum_{m,k} \int_{T_{m,k}} \left(C4^{-m} \int_{U_{(m,k)}} |f(w)|^p \, dA(w) \right)^{\frac{q}{p}} \\
&= \sum_{m,k} \mu\left(T_{m,k}\right) \left(C4^{-m} \int_{U_{(m,k)}} |f(w)|^p \, dA(w) \right)^{\frac{q}{p}}.
\end{aligned}$$

From the hypothesis $\mu\left(T_{m,k}\right) \leq 2^{-2mq/p}$, and then we obtain

$$\int_{\mathbb{D}} |f(z)|^q \, d\mu \leq C \sum_{m,k} \left(\int_{U_{(m,k)}} |f(z)|^p \, dA(z) \right)^{\frac{q}{p}}.$$

A geometric argument shows that for any fixed pair (m_0, k_0) the set T_{m_0,k_0} intersects $U_{(m,k)}$ for at most $N = 300$ choices of the pairs (m, k). Then we obtain

$$\begin{aligned}
\int_{\mathbb{D}} |f(z)|^q(z) \, d\mu &\leq C \sum_{m,k} \left(\int_{U_{(m,k)}} |f(z)|^p \, dA(z) \right)^{\frac{q}{p}} \\
&= C \left(\sum_{m,k} \sum_{m_0,k_0} \int_{T_{m_0,k_0} \cap U_{(m,k)}} |f(z)|^p \, dA(z) \right)^{\frac{q}{p}} \\
&= C \left(\sum_{m_0,k_0} \sum_{m,k} \int_{T_{m_0,k_0} \cap U_{(m,k)}} |f(z)|^p \, dA(z) \right)^{\frac{q}{p}} \\
&\leq \left(N \sum_{m_0,k_0} \int_{T_{m_0,k_0}} |f(z)|^p \, dA(z) \right)^{\frac{q}{p}} \\
&= CN^{q/p} \|f\|_{A^p}^q
\end{aligned}$$

□

Luecking [16, 18] developed a different method to obtain characterization of the q-Carleson measures for the space A^p in the case $p \leq q$ which bring into play the pseudohyperbolic metric.

Theorem 8. *Suppose that $0 < p < q < \infty$ and let μ be a positive Borel measure in $\mathbb{D}$. Then the following three conditions are equivalent:*

(i) μ is a q-Carleson measure for the space A^p.

(ii) *For every* $r \in (0,1)$ *there exists a positive constant* C_r *(which depends on* r *only) such that the inequality*

$$\mu\big(\Delta(a,r)\big) \le C_r \big|\Delta(a,r)\big|^{q/p},$$

holds for all the pseudo-hyperbolic discs of radius r*,* $\Delta(a,r)$ *(*$a \in \mathbb{D}$*).*

(iii) *For a certain* $r \in (0,1)$ *there exist a positive constant* C_r *(which depends on* r *only) such that the inequality*

$$\mu\big(\Delta(a,r)\big) \le C_r \big|\Delta(a,r)\big|^{q/p},$$

holds for all the pseudo-hyperbolic discs of radius r*,* $\Delta(a,r)$ *(*$a \in \mathbb{D}$*).*

Proof. $(i) \to (ii)$: Suppose that $0 < p \le q < \infty$ and μ is a q-Carleson measure for A^p, that is, there exists a positive constant A such that

$$\left(\int_{\mathbb{D}} |f(z)|^q \, d\mu(z)\right)^{1/q} \le A\|f\|_{A^p}, \quad \text{for all } f \in A^p.$$

Take $r \in (0,1)$ and $a \in \mathbb{D}$ and choose the function $f(z) = k_a(z)^{2/p}$, where $k_a(z) = (1-\bar{a}z)^{-2}$ is the Bergman kernel function of $\mathbb{D}$. We have

$$\|f\|_p^q = \|k_a\|_2^{2\frac{q}{p}} = (1-|a|^2)^{2\frac{q}{p}}$$

While

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|^q d\,\mu = \int_{\mathbb{D}} |k_a(z)|^{2\frac{q}{p}} d\,\mu \ge \int_{\Delta(a,r)} |k_a(z)|^{2\frac{q}{p}} d\,\mu \\ \ge \mu\big(\Delta(a,r)\big) \min_{z\in\Delta(a,r)} |k_a(z)|^{2\frac{q}{p}} = \mu\big(\Delta(a,r)\big)\left(\frac{1-r|a|}{1-|a|^2}\right)^{4\frac{q}{p}}\end{aligned}$$

Thus in view of (i)

$$\begin{aligned}\mu\big(\Delta(a,r)\big) &\le C\|f\|_p^q \left(\frac{1-r|a|}{1-|a|^2}\right)^{4\frac{q}{p}} = C\left(\frac{1-|a|^2}{(1-r|a|)^2}\right)^{2\frac{q}{p}} \\ &= C\left(\frac{1-|a|^2}{(1-r|a|)^2}\right)^2 |\Delta(a,r)|^{\frac{q}{p}} \\ &\le \frac{4C}{r^2(1-r)^2}|\Delta(a,r)|^{\frac{q}{p}}.\end{aligned}$$

The implication $(ii) \to (iii)$ is trivial.

$(iii) \to (i)$: Suppose now that there exists $r \in (0,1)$ and $C = C_r > 0$ such that

$$\mu\big(\Delta(a,r)\big) \le C_r \big|\Delta(a,r)\big|^{q/p}, \quad \text{for all } a \in \mathbb{D}.$$

Set $R = \frac{1}{2}(1+r)$ and let us use Lemma 3 to obtain a sequence $\{z_n\} \in \mathbb{D}$ such that $\mathbb{D} = \cup \Delta(z_n, r)$ and a positive integer N such that:

$$\text{Every point in } \mathbb{D} \text{ belongs to at most } N \text{ of the discs } \Delta(z_n, R). \tag{5.3}$$

Take $f \in A^p$. We have

$$\int_{\mathbb{D}} |f(z)|^q d\,\mu \le \sum_{k=1}^{\infty} \int_{\Delta(z_k,r)} |f(z)|^q \, d\mu. \tag{5.4}$$

But Lemma 4 and the hypothesis (iii) imply

$$\begin{aligned}
\int_{\Delta(z_k,r)} |f(z)|^q \, d\mu &= \int_{\Delta(z_k,r)} |f(z)|^{p\frac{q}{p}} \, d\mu \\
&\le C\,\mu\left(\Delta(z_k,r)\right) \left(\frac{1}{|\Delta(z_k,r)|} \int_{\Delta(z_k,R)} |f(z)|^p \, dA(z) \right)^{q/p} \\
&\le C \left(\int_{\Delta(z_k,R)} |f(z)|^p \, dA(z) \right)^{q/p}.
\end{aligned}$$

Then, using (5.4) and (5.3) we obtain

$$\begin{aligned}
\int_{\mathbb{D}} |f(z)|^q d\,\mu &\le C \sum_{k=1}^{\infty} \left(\int_{\Delta(z_k,R)} |f(z)|^p \, dA(z) \right)^{q/p} \\
&\le C\, N^{q/p} \left(\int_{\mathbb{D}} |f(z)|^p \, dA(z) \right)^{q/p} \\
&= C\, N^{q/p} \|f\|_{A^p}^q.
\end{aligned}$$

This finishes the proof. □

Putting together Theorem 7 and Theorem 8 we deduce the following.

Theorem 9. *Suppose that $0 < p < q < \infty$ and let μ be a positive Borel measure in $\mathbb{D}$. Then the following three conditions are equivalent:*

(i) μ is a q-Carleson measure for the space A^p.

(ii) μ is a $(2q)/p$-Carleson measure.

(iii) For every $r \in (0,1)$ there exists a positive constant C_r (which depends on r only) such that the inequality

$$\mu\big(\Delta(a,r)\big) \le C_r \big|\Delta(a,r)\big|^{q/p},$$

holds for all the pseudo-hyperbolic discs of radius r, $\Delta(a,r)$ $(a \in \mathbb{D})$.

(iv) For a certain $r \in (0,1)$ there exist a positive constant C_r (which depends on r only) such that the inequality

$$\mu\big(\Delta(a,r)\big) \le C_r \big|\Delta(a,r)\big|^{q/p},$$

holds for all the pseudo-hyperbolic discs of radius r, $\Delta(a,r)$ $(a \in \mathbb{D})$.

In particular, we have that conditions (ii) and (iii) are equivalent. This equivalence is not entirely trivial but it can be verified by a direct geometric argument:

Given $r \in (0,1)$ and a Carleson box S of side-length h, one can construct a pseudohyperbolic disc $\Delta(a,r)$ contained in S with area $|\Delta(a,r)| \ge c_r|S|$. From this if follows that Hastings' condition (ii) implies (iii). The other implication follows from the fact that a given Carleson box can be covered by pseudohyperbolic discs of radius r that intersect at most a bounded number of times as in Lemma 3. We omit the details.

5.2 The case $q < p$

The q-Carleson measures for the space A^p $(q < p)$ were characterized by Luecking [17].

Theorem 10. *Suppose then $0 < q < p < \infty$ and let μ be a positive Borel measure in $\mathbb{D}$. Then the following two conditions are equivalent:*

(a) μ is a q-Carleson measure for the Bergman space A^p

(b) For any given $r \in (0,1)$ t he function $z \to \frac{\mu(\Delta(z,r))}{(1-|z|^2)^{-2}}$ belongs $L^{\frac{p}{p-q}}(dA)$.

The proof of Theorem 10 given in [17] used a result of Amar [1] about interpolation sequences for Bergman spaces. Later on Luecking [20] obtained a new proof of his result using a part of Amar's result and Khinchine's inequality. We shall not present the proof here.

Chapter 6

Some examples and applications

Clearly the Lebesgue measure in $\mathbb{D}$ is a Carleson measure but there are many other examples. In this chapter we shall present several distinct explicit examples of Carleson measures for Hardy spaces and Bergman spaces and we shall also show how a number of classical results can be deduced from these examples.

6.1 Some discrete Carleson measures

(1) Suppose that $\{a_n\}_{n=1}^{\infty}$ is an exponential sequence in $\mathbb{D}$, say

$$(1-|a_{n+1}|) \leq \lambda(1-|a_n|), \quad n \geq 1, \text{ with } 0<\lambda<1. \tag{6.1}$$

Set

$$\mu = \sum_{n=1}^{\infty}(1-|a_n|)\delta_{a_n}.$$

Since any exponential sequence is uniformly separated, Carleson theorem implies that μ is a Carleson measure. It is easy to this fact directly in this case. Indeed, take a Carleson box $S = S_{\theta,h}$. We have

$$\mu(S) = \sum_{a_n \in S}(1-|a_n|) \leq \sum_{|a_n| \geq 1-h}(1-|a_n|) = \sum_{1-|a_n| \leq h}(1-|a_n|).$$

Now, (6.1) implies

$$(1-|a_j|) \leq \lambda^{j-k}(1-|a_k|), \quad j \geq k,$$

and then we have

$$\mu(S) \leq \sum_{1-|a_n| \leq h}(1-|a_n|) \leq \sum_{j=k_0}^{\infty}\lambda^{j-k_0}(1-|a_{k_0}|) = C_\lambda(1-|a_{k_0}|),$$

where k_0 is the first integer k so that $1-|a_{k_0}| \leq h$. Thus, we have $\mu(S) \leq C_\lambda h$. This shows that μ is a Carleson measure as claimed.

(2) Let s be a fixed number with $s > 1$ and set

$$a_j = 1 - j^{-s}, \quad j = 1, 2, 3, \ldots,$$

and

$$\mu = \sum_{j=1}^{\infty} (1 - |a_j|)\delta_{a_j} = \sum_{j=1}^{\infty} j^{-s}\delta_{a_j}.$$

Take $0 < h < 1$ and $\theta = -h/2$. For the Carleson box $S = S_{\theta,h}$ we have

$$\mu(S) = \sum_{a_n \in S} (1 - |a_n|) = \sum_{1-|a_n| \leq h} (1 - |a_n|) = \sum_{n \geq h^{-1/s}} \frac{1}{n^s} \asymp h^{(s-1)/s}.$$

Thus we see that μ is not a Carleson measure and hence we deduce that for any $p \in (1, \infty)$ the operator T_p defined by

$$T_p(f) = \{(1 - |a_n|)^{1/p} f(a_n)\}$$

which was considered in Chapter 3, does not map H^p into ℓ^p even though the sequence $\{a_n\}$ is a Blaschke sequence.

6.2 Carleson measures and the Fejér-Riesz inequality

Let $d\mu = |dz|$ on a radius, that is, with $\theta \in \mathbb{R}$ fixed, then for any Borel set $S \subset \mathbb{D}$, μ is given by

$$\mu(S) = \text{length}\left(S \cap [0, e^{i\theta}]\right).$$

It is clear that μ is a Carleson measure and by Theorem 3 that $H^p \subset L^p(d\mu)$. This can be written as follows:

There exists a positive constant C_p such that

$$\int_0^1 |f(re^{i\theta})|^p \, dr \leq C_p \|f\|_{H^p}^p, \quad f \in H^p, \ \theta \in \mathbb{R}.$$

Thus we have obtained the classical Fejér-Riesz inequality, except for the exact value of the constant, (see [5, Theorem 3.13]).

6.3 Carleson measures and embedding theorems

(a) The Lebesgue area measure in $\mathbb{D}$ is a 2-Carleson measure. Indeed, for the Carleson box $S_{\theta,h}$ we have

$$|S_{\theta,h}| \asymp \int_{1-h}^{1} \int_{\theta}^{\theta+h} dA(z) \asymp h^2.$$

Thus, using Theorem 3, we see that the Lebesgue area measure dA is a $2p$-Carleson measure for H^p for all $p \in (0,\infty)$. This gives the embedding

$$H^p \subset A^{2p}, \quad 0 < p < \infty$$

which we mentioned in Chapter 1.

Similarly, a simple computation shows that if $\alpha > 1$ then the measure μ defined by

$$d\mu(z) = (1-|z|)^{\alpha-2} dA(z)$$

is an α-Carleson measure. Taking $0 < p \leq q < \infty$ and $\alpha = q/p$ and using Theorem 3 we deduce the following classical result of Hardy-Littlewood:

$$f \in H^p \;\Rightarrow\; \int_{\mathbb{D}} (1-|z|)^{\frac{q}{p}-2} |f(z)|^q \, dA(z) < \infty, \; 0 < p < q < \infty.$$

(b) The computation in (a) shows that if $\alpha \geq 1$ then the measure μ defined by

$$d\mu(z) = (1-|z|)^{2\alpha-2} dA(z)$$

is a 2α-Carleson measure. Taking $0 < p \leq q < \infty$ and $\alpha = q/p$, and using Hastings' theorem, we deduce that

$$f \in A^p \;\Rightarrow\; \int_{\mathbb{D}} (1-|z|)^{2\frac{q}{p}-2} |f(z)|^q \, dA(z),$$

that is A^p is contained in the weighted Bergmam space A^q_β with $\beta = 2\frac{q}{p} - 2$.

6.4 $BMOA$ and Carleson measures

If $f \in L^1(\mathbb{T})$ and I is an interval of $\mathbb{T}$, we let $m_I(f)$ denote the mean of f over the interval I, that is,

$$m_I(f) = \frac{1}{|I|} \int_I f(e^{i\theta}) \, d\theta.$$

The mean oscillation of f over I is

$$m_I\left(|f - m_I(f)|\right) = \frac{1}{|I|}\int_I |f(e^{i\theta}) - m_I(f)|\, d\theta.$$

We say that f has bounded mean oscillation or that $f \in BMO$ if the supremum of these quantities as I ranges over all intervals $I \subset \mathbb{T}$ is finite:

$$f \in BMO \;\Leftrightarrow\; f \in L^1(\mathbb{T}) \text{ and } \|f\|_\star = \sup_I \frac{1}{|I|}\int_I |f(e^{i\theta}) - m_I(f)|\, d\theta < \infty.$$

Actually, BMO functions were introduced by John and Nirenberg [13], in the context of of functions defined in cubes of $\mathbb{R}^n$ and they applied them to smoothness problems in partial differential equations. However they have achieved an extremely important role in the theory of harmonic and analytic functions. The first connection comes from the fact that the conjugation operator applies $L^\infty_{\mathbb{R}}$ into $BMO_{\mathbb{R}}$. This was first noticed by Spane [22] and Stein [23]. Of course, the most important result in this line is Fefferman's duality theorem: $(H^1_{\mathbb{R}})^\star = BMO$ [7, 8].

Even though a BMO function need not be bounded, it cannot grow too fast. Even more, John an Nirenberg proved that BMO functions satisfy a condition of exponential integrability.

Theorem 11. *There exist two positive constants K and β such that if $f \in BMO$ is not constant and I is an interval of $\mathbb{T}$ then*

$$\frac{1}{|I|}\left|\{e^{i\theta} \in I : |f(e^{i\theta}) - m_I(f)| > \lambda\}\right| \le K e^{-\beta\frac{\lambda}{\|f\|_\star}}, \quad \textit{for every } \lambda > 0.$$

John-Nirenberg's theorem does imply that BMO-functions satisfy very strong integrability conditions and it allows us to find equivalent norms for BMO. In particular, we have that, for all $p \ge 1$, $BMO \subset L^p(\mathbb{T})$ and

$$\|f\|_\star \asymp \|f\|_{\star,p} \overset{\text{def}}{=} \sup_I \left(\frac{1}{|I|}\int_I |f - m_I(f)|^p\, d\theta\right)^{1/p}.$$

From now on, we shall restrict to the "analytic-BMO-space", $BMOA$, which consists of those functions $f \in H^1$ whose boundary values function $f(e^{i\theta})$ belongs to BMO. It is a Banach space with any of the (equivalent) norms

$$\|f\|_{\star,p} = |f(0)| + \sup_I \left(\frac{1}{|I|}\int_I |f(e^{i\theta}) - m_I(f)|^p\, d\theta\right)^{1/p}.$$

We remark that Fefferman's duality theorem can be stated as follows:

$$\left(H^1\right)^\star = BMOA.$$

We mention [9],[10] and [21] for the theory of BMO and $BMOA$-functions.

A key fact underlying the importance of BMO-functions in the theory of analytic functions in $\mathbb{D}$ is the conformal invariance of $BMOA$:

Suppose that $0 < p < \infty$. A function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to $BMOA$ if and only if the family $\{f \circ \varphi_a - f(a) : a \in \mathbb{D}\}$ is a bounded set of H^p. Furthermore, if we define

$$\|f\|_{BMOA_p} = |f(0)| + \sup_{a\in\mathbb{D}} \|f \circ \varphi_a - f(a)\|_{H^p}, \tag{6.2}$$

we have that $\|\cdot\|_{BMOA_p}$ s a norm in $BMOA$ equivalent to any of the above defined $\|\cdot\|_\star$-norms.

Using the characterization of $BMOA$ in terms of the $BMOA_2$ norm and the Littlewood-Paley identties (Theorem 1) we obtain another characterization of $BMOA$-functions.

Theorem 12. *Let f be an analytic function in $\mathbb{D}$. Then the following conditions are equivalent:*

(i) $f \in BMOA$.

(ii) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g(a,z)\, dA(z) < \infty$.

(iii)

$$\begin{aligned}&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \left(1 - |\varphi_a(z)|^2\right) dA(z)\\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \frac{(1-|z|^2)(1-|a|^2)}{|1-\overline{a}z|^2}\, dA(z) < \infty.\end{aligned}$$

Here,

$$g(a,z) = \log\left|\frac{1}{\varphi_a(z)}\right| = \log\left|\frac{1-\overline{a}z}{a-z}\right|, \quad a, z \in \mathbb{D},$$

is the Green function of $\mathbb{D}$.

Moreover,

$$\|f\|_{BMOA_2} = |f(0)| + \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^2 g(a,z)\, dA(z)\right)^{1/2}$$

and the function

$$f \;\mapsto\; |f(0)| + \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^2 \left(1 - |\varphi_a(z)|^2\right) dA(z)\right)^{1/2}$$

is a norm in $BMOA$ which is equivalent to the $BMOA_2$ norm (and, hence to any of the equivalent norms we have defined so far).

Using this result and Carleson theorem (Theorem 3 for $p = q = 2$), we finally obtain the characterization of $BMOA$ in terms of Carleson measures.

Theorem 13. *Let f be an analytic function in $\mathbb{D}$. Then the three following conditions are equivalent:*

(i) $f \in BMOA$.

(ii) If μ_f is the measure in $\mathbb{D}$ defined by

$$d\mu_f(z) = (1 - |z|^2)|f'(z)|^2 \, dA(z)$$

then μ_f is a Carleson measure.

(iii) If ν_f is the measure in $\mathbb{D}$ defined by

$$d\nu_f(z) = \left(\log \frac{1}{|z|}\right) |f'(z)|^2 \, dA(z)$$

then ν_f is a Carleson measure.

Moreover,

$$\|f\|_{BMOA_2} \approx |f(0)| + N(\mu_f)^{1/2} \approx |f(0)| + N(\nu_f)^{1/2}.$$

We remark that this result is a key ingredient in the proof of Fefferman's duality theorem: $(H^1)^\star = BMOA$.

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